

# SUPERFORMS, TROPICAL COHOMOLOGY AND POINCARÉ DUALITY

PHILIPP JELL, KRISTIN SHAW, JASCHA SMACKA

ABSTRACT. We establish a canonical isomorphism between two bigraded cohomology theories for polyhedral spaces: Dolbeault cohomology of superforms and tropical cohomology. Furthermore, we prove Poincaré duality for cohomology of tropical manifolds, which are polyhedral spaces locally given by Bergman fans of matroids.

December 24, 2015

## CONTENTS

1. Introduction	1
2. Superforms	3
2.1. Superforms on polyhedral subspaces of tropical affine space	3
2.2. Polyhedral spaces	7
3. Comparison of cohomologies	9
3.1. Tropical cohomology	10
3.2. Dolbeault cohomology of superforms	13
3.3. Equivalence of cohomologies	16
4. Poincaré duality	17
4.1. Integration of superforms	18
4.2. Poincaré duality for tropical manifolds	21
References	28

## 1. INTRODUCTION

In [Lag12], Lagerberg introduced superforms on  $\mathbb{R}^r$ . Superforms are bigraded real-valued differential forms with differential operators  $d'$ ,  $d''$  and  $d$ , analogous to the differential operators  $\partial$ ,  $\bar{\partial}$  and  $d$  on complex differential forms. Recently, superforms restricted to tropicalizations were used by Chambert-Loir and Ducros to construct real-valued differential forms on analytic spaces in the sense of Berkovich [CLD12]. Superforms have also been used to provide a non-Archimedean analytic description of heights by Gubler and Künnemann [GK14].

A Poincaré lemma with respect to the differential operators  $d'$  and  $d''$  for superforms on polyhedral complexes in  $\mathbb{R}^r$  and Berkovich spaces was proven in [Jel15]. Here we will consider the cohomology with respect to the operator  $d''$ . We call this the Dolbeault cohomology of superforms since the operator  $d''$  behaves analogously to the operator  $\bar{\partial}$  for complex differential forms.

Tropical cohomology of a polyhedral complex, as introduced by the authors of [IKMZ], is the cohomology of singular cochains with non-constant coefficients. The coefficient systems are determined by the geometry of the complex (cf. [MZ13] or Section 3.1 for definitions). Via

the tropicalization procedure, this cohomology theory can be related to the Hodge theory of projective varieties. For example, under suitable conditions, the ranks of the tropical cohomology groups can be related to the Hodge numbers of projective varieties (cf. [IKMZ], also [MZ13, Remark 2.4], and [BIMS15, Theorem 7.34]).

Our first goal here is to prove that Dolbeault cohomology of superforms and tropical cohomology on a polyhedral space are canonically isomorphic. Before doing so, we extend the theory of superforms to polyhedral complexes contained in a partial compactification of  $\mathbb{R}^r$  that arises in tropical geometry,  $\mathbb{T}^r := [-\infty, \infty)^r$ . Polyhedral spaces are topological spaces which are equipped with an atlas of charts to polyhedral complexes in  $\mathbb{T}^r$  (cf. Definition 2.18). We will sometimes restrict our considerations to polyhedral spaces which are regular at infinity (c.f. once again Definition 2.18). For a polyhedral space  $X$ , we obtain complexes of sheaves of superforms  $(\mathcal{A}_X^{p,\bullet}, d'')$  by glueing the sheaves on open subsets.

Let  $X$  be a polyhedral space equipped with a face structure (cf. Definition 3.2). In Subsection 3.1, we recall the definition of tropical cohomology groups  $H_{\text{trop}}^{p,q}(X)$  and extend this to tropical cohomology groups with compact support  $H_{\text{trop},c}^{p,q}(X)$  using singular chains. We call the cohomology groups of the complexes  $(\mathcal{A}_X^{p,\bullet}(X), d'')$  the Dolbeault cohomology of superforms and denote the  $q$ -th cohomology group of this complex by  $H_{d''}^{p,q}(X)$ . Similarly, we write  $H_{d'',c}^{p,q}(X)$  for the cohomology of global sections with compact support (cf. Definition 2.21). The first theorem relates the Dolbeault cohomology of superforms and tropical cohomology.

**Theorem 1.** *Let  $X$  be a polyhedral space which is regular at infinity and equipped with a face structure. Then there are canonical isomorphisms*

$$\begin{aligned} H_{\text{trop}}^{p,q}(X) &\cong H_{d''}^{p,q}(X) \text{ and} \\ H_{\text{trop},c}^{p,q}(X) &\cong H_{d'',c}^{p,q}(X). \end{aligned}$$

In order to show this we first establish that, for every  $p$ , the complex  $\mathcal{A}_X^{p,\bullet}$  is an acyclic resolution of certain sheaves denoted  $\mathcal{L}_X^p$  on  $X$ . Tropical cohomology was already shown to be equivalent to the cohomology of constructible sheaves, denoted  $\mathcal{F}_X^p$ , in [MZ13, Proposition 2.8]. Comparing explicit descriptions of these sheaves on a basis of the topology, we show that  $\mathcal{L}_X^p$  and  $\mathcal{F}_X^p$  are isomorphic, which implies the above theorem. In fact, the sheaves  $\mathcal{F}_X^p$  are defined for a polyhedral space  $X$  even in the absence of a face structure. Thus the above argument relates Dolbeault cohomology of superforms with the cohomology of the sheaves  $\mathcal{F}_X^p$  from [MZ13] for general polyhedral spaces (cf. Remark 3.19). The argument proving the equivalence of Dolbeault and tropical cohomologies also applies to the respective cohomologies with compact support.

Secondly, we prove a version of Poincaré duality for tropical manifolds. For  $X$  an  $n$ -dimensional tropical space (cf. Definition 4.6), there is a map

$$\text{PD} : H_{d''}^{p,q}(X) \rightarrow H_{d'',c}^{n-p,n-q}(X)^*,$$

which we call the Poincaré duality map. This map is induced by integration of superforms (cf. Definition 4.8), and thus is similar to the integration pairing on cohomology of a complex manifold. The fact that the Poincaré duality map on spaces of superforms descends to cohomology when  $X$  is a tropical space follows from a version of Stokes' theorem (cf. Theorem 4.7).

Tropical manifolds are tropical spaces with the extra condition that they are locally modeled on Bergman fans of matroids (cf. [Stu02], [MR], [Sha11]). These matroidal fans can arise as tropicalizations of linear spaces, however they are much more general and may even have no algebraic counterpart. Despite perhaps being far from smooth objects in the algebraic or differentiable sense, tropical manifolds exhibit many properties analogous to smooth spaces (cf. [Sha11]). Establishing Poincaré duality for the cohomology of these spaces provides another instance of this phenomenon.

**Theorem 2.** *If  $X$  is an  $n$ -dimensional tropical manifold then the Poincaré duality map is an isomorphism for all  $p, q$ .*

As in the proof of the classical case, Poincaré duality is first established for the local models of tropical manifolds, which, as mentioned above, are matroidal fans. This is done in Propositions 4.19 and 4.22. The main ingredient in the proof of the local case is a recursive description of matroidal fans based on deletion and contraction of matroids, which is equivalent to tropical modifications of the fans (cf. [Sha13b]). Poincaré duality for general tropical manifolds is then established from the local situation via standard methods.

In recent work, Adiprasito, Huh and Katz consider an intersection ring associated to a matroid. For a matroid  $M$ , the graded ring  $A^*(M)$  from [AHK] is shown to satisfy many striking properties in line with the cohomology rings of compact Kähler manifolds, such as Poincaré duality, the Hard Lefschetz theorem and an analogue of the Hodge-Riemann bilinear relations. We expect that this ring is related to the cohomology groups presented here in the following way: For a matroid  $M$ , let  $V$  denote its Bergman fan, then there is a suitable compactification  $\overline{V}$  of  $V$  for which  $H^{k,k}(\overline{V}) \cong A^k(M)$ . Moreover, the product structures on  $H^{*,*}(\overline{V})$  and  $A^*(M)$  should also be isomorphic.

Under suitable conditions on the polyhedral space, tropical cohomology has been seen to have certain properties which one would expect from Hodge structures. For example, the Lefschetz type theorems established in [AB], and the Poincaré duality proved here in the case of tropical manifolds. However, it is already known that the tropical cohomology of tropical manifolds does not in general satisfy a direct translation of the Hodge-Riemann bilinear relations [Sha13a]. Also, an interesting open question is to establish the appropriate condition on a polyhedral space  $X$  so that  $H^{p,q}(X) \cong H^{q,p}(X)$  (cf. [MZ13, Section 5]).

We now outline the presentation of this paper. Section 2 reviews superforms on  $\mathbb{R}^r$  and extends their definition to superforms on  $\mathbb{T}^r = [-\infty, \infty)^r$ . Superforms on  $\mathbb{T}^r$  require compatibility conditions along the strata of this partial compactification of  $\mathbb{R}^r$ . For an open subset of the support of a polyhedral complex in  $\mathbb{T}^r$  we define the space of  $(p, q)$ -superforms and show that this produces a sheaf. This construction is also extended to produce sheaves of superforms on polyhedral spaces. Section 3 recalls the definitions of tropical cohomology and calculates the cohomology of basic open sets (cf. Definition 3.7). It also establishes a Poincaré lemma for the complexes of superforms on a polyhedral space and furthermore computes the sections of  $\mathcal{L}_X^p$  over basic open sets. Following this, the Dolbeault cohomology of superforms and tropical cohomology are shown to be isomorphic (cf. Theorem 3.18). Subsection 4.1 introduces integration and proves Stokes' theorem mentioned above. Finally, Subsection 4.2 is devoted to the proof of Poincaré duality for tropical manifolds (cf. Theorem 4.26).

**Acknowledgements.** The authors would like to thank Walter Gubler, Johann Haas and Klaus Künnemann for comments on a preliminary draft, and also Karim Adiprasito, Grigory Mikhalkin, Johannes Rau and Ilia Zharkov for fruitful discussions.

The second author's research is supported by a postdoctoral research fellowship from the Alexander von Humboldt Foundation. The first and third author would like to thank the collaborative research centre SFB 1085 "Higher Invariants" by the Deutsche Forschungsgemeinschaft for its support.

All authors would like to thank the Graduierten Kolleg "GRK 1692" by the Deutsche Forschungsgemeinschaft for making possible the lecture series by the second author that inspired this collaboration.

## 2. SUPERFORMS

**2.1. Superforms on polyhedral subspaces of tropical affine space.** In this subsection we define bigraded sheaves of superforms on polyhedral complexes in tropical affine space  $\mathbb{T}^r$  (cf. Definition 2.3). We start by recalling the definitions for open subsets of  $\mathbb{R}^r$  from

[Lag12]. After that we extend these to open subsets of  $\mathbb{T}^r$  and to open subsets of polyhedral complexes in  $\mathbb{T}^r$ .

**Definition 2.1.** Let  $U \subset \mathbb{R}^r$  be an open subset. Denote by  $\mathcal{A}^q(U)$  the space of differential forms of degree  $q$  on  $U$ . The space of *superforms of bidegree*  $(p, q)$  on  $U$  is defined as

$$\mathcal{A}^{p,q}(U) := \mathcal{A}^p(U) \otimes_{C^\infty(U)} \mathcal{A}^q(U) = \Lambda^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \mathcal{A}^q(U).$$

If we choose a basis  $x_1, \dots, x_r$  of  $\mathbb{R}$ , following [CLD12] and [Gub13], we formally write a superform  $\alpha \in \mathcal{A}^{p,q}(U)$  as

$$\alpha = \sum_{|K|=p, |L|=q} \alpha_{KL} d'x_K \wedge d''x_L$$

where  $K = \{i_1, \dots, i_p\}$  and  $L = \{j_1, \dots, j_q\}$  are ordered subsets of  $\{1, \dots, r\}$ , the coefficients  $\alpha_{KL} \in C^\infty(U)$  are smooth functions and

$$d'x_K \wedge d''x_L := (dx_{i_1} \wedge \dots \wedge dx_{i_p}) \otimes_{\mathbb{R}} (dx_{j_1} \wedge \dots \wedge dx_{j_q}).$$

There is a differential operator

$$d'' : \mathcal{A}^{p,q}(U) = \Lambda^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \mathcal{A}^q(U) \rightarrow \mathcal{A}^{p,q+1}(U) = \Lambda^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \mathcal{A}^{q+1}(U),$$

given by  $(-1)^p \text{id} \otimes D$ , where  $D$  is the usual differential operator on forms. In coordinates we have

$$\begin{aligned} d'' \left( \sum_{K,L} \alpha_{KL} d'x_K \wedge d''x_L \right) &= \sum_{K,L} \sum_{i=1}^r \frac{\partial \alpha_{KL}}{\partial x_i} d''x_i \wedge d'x_K \wedge d''x_L \\ &:= (-1)^p \sum_{K,L} \sum_{i=1}^r \frac{\partial \alpha_{KL}}{\partial x_i} d'x_K \wedge d''x_i \wedge d''x_L. \end{aligned}$$

There is also a wedge product that is, up to sign, induced by the usual wedge products

$$\begin{aligned} \mathcal{A}^{p,q}(U) \times \mathcal{A}^{p',q'}(U) &\rightarrow \mathcal{A}^{p+p',q+q'}(U) \\ (\alpha, \beta) &\mapsto \alpha \wedge \beta, \end{aligned}$$

which in coordinates is given by

$$\begin{aligned} (\alpha_{KL} d'x_K \wedge d''x_L) \wedge (\beta_{K'L'} d'x_{K'} \wedge d''x_{L'}) &:= \alpha_{KL} \beta_{K'L'} d'x_K \wedge d''x_L \wedge d'x_{K'} \wedge d''x_{L'} \\ &:= (-1)^{p'q} \alpha_{KL} \beta_{K'L'} d'x_K \wedge d'x_{K'} \wedge d''x_L \wedge d''x_{L'}. \end{aligned}$$

If one of  $\alpha, \beta$  has compact support then so does  $\alpha \wedge \beta$ .

Note that we have the usual Leibniz formula

$$d''(\alpha \wedge \beta) = d''\alpha \wedge \beta + (-1)^{p+q} \alpha \wedge d''\beta.$$

**Remark 2.2.** There are also differential operators  $d' := D \otimes \text{id}$  and  $d := d' + d''$ , which are not considered in this paper. It is easy to see that the theories for  $d'$  and  $d''$  are symmetric up to sign. We choose to consider the operator  $d''$ , since it produces the same cohomology as tropical cohomology. The cohomology of the operator  $d'$  is isomorphic to that of  $d''$  up to switching the bigrading.

Let  $\mathbb{T} = [-\infty, \infty)$  and equip it with the topology of a half open interval. Then  $\mathbb{T}^r$  is equipped with the product topology. We write  $[r] := \{1, \dots, r\}$ .

**Definition 2.3.** The *sedentarity* of a point  $x \in \mathbb{T}^r$  is the subset  $\text{sed}(x) \subset [r]$  consisting of coordinates of  $x$  which are  $-\infty$ .

The space  $\mathbb{T}^r$  is naturally stratified by the sedentarity of points. For  $I \subset [r]$  set

$$\mathbb{R}_I^r = \{x \in \mathbb{T}^r \mid x_i = -\infty \text{ if and only if } i \in I\}.$$

Clearly,  $\mathbb{R}_I^r \cong \mathbb{R}^{r-|I|}$ . As a convention throughout, for a subset  $S \subset \mathbb{T}^r$  we denote  $S_I := S \cap \mathbb{R}_I^r$ .

Moreover, for  $J \subset I$  there is a canonical projection  $\pi_{IJ} : \mathbb{R}_J^r \rightarrow \mathbb{R}_I^r$ . Coordinate-wise the map  $\pi_{IJ}$  sends  $x_i$  to  $-\infty$  if  $i \in I$  and to  $x_i$  otherwise.

**Definition 2.4.** Let  $U \subset \mathbb{T}^r$  be an open subset. A  $(p, q)$ -superform  $\alpha$  on  $U$  is given by a collection of superforms  $(\alpha_I)_I$  such that,

- i)  $\alpha_I \in \mathcal{A}^{p,q}(U_I)$  for all  $I$ ,
- ii) for each point  $x \in U \subset \mathbb{T}^r$  of sedentarity  $I$ , there exists a neighborhood  $U_x$  of  $x$  contained in  $U$  such that for each  $J \subset I$  the projection satisfies  $\pi_{IJ}(U_{x,J}) = U_{x,I}$  and  $\pi_{IJ}^*(\alpha_I|_{U_{x,I}}) = \alpha_J|_{U_{x,J}}$ .

We denote the space of  $(p, q)$ -superforms on  $U$  by  $\mathcal{A}^{p,q}(U)$ . Note that a superform in  $\mathcal{A}^{0,0}(U)$  defines a continuous function on  $U$ . We will thus sometimes refer to  $(0, 0)$ -superforms as smooth functions.

Condition ii) of Definition 2.4 will be referred to as the *condition of compatibility* of superforms along strata. Suppose  $U \subset \mathbb{T}^r$  is an open set whose points have a unique maximal sedentarity  $I$  and  $\alpha \in \mathcal{A}^{p,q}(U)$ . If for each  $J \subset I$  we have  $\pi_{IJ}^* \alpha_I = \alpha_J$ , then we say that  $\alpha$  is *determined by  $\alpha_I$  on  $U$* . Notice that the condition of compatibility along strata implies that each  $x \in U$  has an open neighborhood  $U_x$  such that  $\alpha|_{U_x}$  is determined by  $(\alpha|_{U_x})_{\text{sed}(x)}$  on  $U_x$ .

If  $U \subset \mathbb{T}^r$  and  $\alpha = (\alpha_I)_I \in \mathcal{A}^{p,q}(U)$  is a superform, define  $d''\alpha$  to be given by the collection  $(d''\alpha_I)_I$ . Pullbacks along the projections  $\pi_{IJ}$  commute with  $d''$ , therefore  $d''\alpha$  is a superform in  $\mathcal{A}^{p,q+1}(U)$ .

If also  $\beta = (\beta_I)_I \in \mathcal{A}^{p',q'}(U)$ , then we define  $\alpha \wedge \beta := (\alpha_I \wedge \beta_I)_I \in \mathcal{A}^{p+p',q+q'}(U)$ . This is indeed a superform on  $U$ , since the pullbacks along the projections commute with the wedge product.

**Remark 2.5.** Notice that there is a natural isomorphism  $J : \mathcal{A}^{p,q}(U) \rightarrow \mathcal{A}^{q,p}(U)$ , which, up to sign, maps  $d'x_K \otimes d''x_L$  to  $d'x_L \otimes d''x_K$  for  $U \subset \mathbb{T}^r$  (cf. [CLD12, Section (1.2.5)]). This is clear when  $U \subset \mathbb{R}^r$ . When  $U$  contains points of non-empty sedentarity the map  $J$  preserves the condition of compatibility on the boundary strata. It is easy to see that this so called *involution* morphism will still be defined for the spaces of superforms on polyhedral subspaces and spaces defined in later sections. In the theory of tropical cohomology, (cf. Subsection 3.1), such an involution does not exist on the chain level.

**Example 2.6.** Consider an open neighborhood  $U$  of  $-\infty \in \mathbb{T}$ . For a  $(p, q)$ -superform  $\alpha \in \mathcal{A}^{p,q}(U)$  with  $\max(p, q) = 1$ , by the condition of compatibility of superforms along strata, there must exist a smaller neighborhood  $U' \subset U$  of  $-\infty$  such that  $\alpha$  is zero on  $U'$ .

Similarly, a  $(0, 0)$ -superform on  $U$  must be a constant function in some neighborhood of  $-\infty$ .

In the next lemma we use upper indexing to avoid confusion with the notation for the sedentarity of sets.

**Lemma 2.7.** Let  $U \subset \mathbb{T}^r$  be an open subset and  $(U^l)_{l \in L}$  an open cover of  $U$ . Then there exist a countable, locally finite cover  $(V^k)_{k \in K}$  of  $U$ , a collection of non-negative smooth functions  $(f^k : V^k \rightarrow \mathbb{R})_{k \in K}$  with compact support and a map  $s : K \rightarrow L$  such that  $V^k \subset U^{s(k)}$  for every  $k \in K$ , and  $\sum_{k \in K} f^k \equiv 1$ . Such a family is called a *partition of unity subordinate to the cover  $(U^l)_{l \in L}$* .

*Proof.* We first show that for any  $x \in \mathbb{T}^r$  and any open neighborhood  $x \in V$  there exists a non-negative function  $f \in \mathcal{A}^{0,0}(\mathbb{T}^r)$  and a neighborhood  $V'$  of  $x$  such that  $f|_{V'} \equiv 1$  and  $\text{supp}(f) \subset V$  is compact. This is clear if  $r = 1$ . Otherwise, a basis of open neighborhoods of  $x$  is given by products of open sets in  $\mathbb{T}^1$ , thus we may assume  $V$  to be of that form. Then taking functions  $f^i$  neighborhood on  $\mathbb{T}^1$  with the above property for every  $i \in [r]$  and defining  $f(x_1, \dots, x_r) = \prod f^i(x_i)$  gives the desired function.

The general theorem then follows from standard arguments, see for instance the proof in [War83, Theorem 1.11].  $\square$

**Definition 2.8.** A *polyhedron* in  $\mathbb{R}^r$  is a subset defined by a system of affine (non-strict) inequalities. A *face* of a polyhedron  $\sigma$  is a polyhedron which is obtained by turning some of the defining inequalities of  $\sigma$  into equalities. For conventions of convex geometry we follow [Gub12, Appendix A].

A *polyhedron* in  $\mathbb{T}^r$  is the closure of a polyhedron in  $\mathbb{R}_I^r \cong \mathbb{R}^{r-|I|} \subset \mathbb{T}^r$  for some  $I \subset [r]$ . A *face* of a polyhedron  $\sigma$  in  $\mathbb{T}^r$  is the closure of a face of  $\sigma \cap \mathbb{R}_J^r$  for some  $J \subset [r]$ . A *polyhedral complex*  $\mathcal{C}$  in  $\mathbb{T}^r$  is a finite set of polyhedra in  $\mathbb{T}^r$ , satisfying the following properties:

- i) For a polyhedron  $\sigma \in \mathcal{C}$ , if  $\tau$  is a face of  $\sigma$  (denoted  $\tau \prec \sigma$ ) we have  $\tau \in \mathcal{C}$ .
- ii) For two polyhedra  $\sigma, \tau \in \mathcal{C}$  the intersection  $\sigma \cap \tau$  is a face of both  $\sigma$  and  $\tau$ .

The maximal polyhedra, with respect to inclusion, are called facets. The support of a polyhedral complex  $\mathcal{C}$  is the union of all its polyhedra and is denoted  $|\mathcal{C}|$ . If  $X = |\mathcal{C}|$ , then  $X$  is called a *polyhedral subspace* of  $\mathbb{T}^r$  and  $\mathcal{C}$  is called a *polyhedral structure* on  $X$ .

The relative interior of a polyhedron  $\sigma$  is denoted  $\text{int}(\sigma)$ . We use the notation  $\mathcal{C}_I$  for the polyhedral complex in  $\mathbb{R}_I^r$  obtained by intersecting all polyhedra of  $\mathcal{C}$  with  $\mathbb{R}_I^r$ . Notice that  $|\mathcal{C}_I| = |\mathcal{C}|_I$ .

**Definition 2.9.** Let  $\mathcal{C}$  be a polyhedral complex in  $\mathbb{T}^r$  and  $\sigma \in \mathcal{C}$ . Let  $x \in \sigma$  and  $I := \text{sed}(x)$ . Then  $\sigma_I := \sigma \cap \mathbb{R}_I^r$  is a polyhedron in  $\mathbb{R}_I^r$ . Define *the tangent space of  $\sigma$  at  $x$*  to be  $\mathbb{L}(\sigma, x) := \mathbb{L}(\sigma_I) \subset \mathbb{R}_I^r$ , where  $\mathbb{L}(\sigma_I)$  is the tangent space to  $\sigma_I$  at any point in its relative interior.

The evaluation of a  $(p, q)$ -superform  $\alpha$  at a collection of vectors  $v_1, \dots, v_p, w_1, \dots, w_q \in \mathbb{L}(\sigma, x)$  is denoted  $\langle \alpha_I(x); v_1, \dots, v_p, w_1, \dots, w_q \rangle$ .

Let  $U \subset \mathbb{T}^r$  be an open subset and  $\alpha = (\alpha_I)_I \in \mathcal{A}^{p,q}(U)$ . For  $v \in \mathbb{R}^r$  and  $s \in [p]$  we define *the contraction of  $\alpha$  with  $v$  in the  $s$ -th component*  $\langle \alpha, v \rangle_s$  to be given by the collection  $(\langle \alpha_I, \pi_{I,\emptyset}(v) \rangle_s)_I$ . Here  $\langle \alpha_I, \pi_{I,\emptyset}(v) \rangle_s \in \mathcal{A}^{p-1,q}(U_I)$  is the contraction of  $\alpha_I$  with  $\pi_{I,\emptyset}(v)$  in the sense of multilinear forms (cf. [Gub13, 2.6]). We obtain a well defined form in  $\mathcal{A}^{p-1,q}(U)$ .

Next we consider the restriction of bigraded superforms to polyhedral complexes in  $\mathbb{T}^r$ .

**Definition 2.10.** Let  $\mathcal{C}$  be a polyhedral complex in  $\mathbb{T}^r$  and  $\Omega \subset |\mathcal{C}|$  an open subset, then a *superform of bidegree  $(p, q)$  on  $\Omega$*  is given by an open subset  $U \subset \mathbb{T}^r$  and a superform  $\alpha \in \mathcal{A}^{p,q}(U)$  such that  $U \cap |\mathcal{C}| = \Omega$ . Two such pairs  $(U, \alpha)$  and  $(U', \alpha')$  are equivalent if for any  $\sigma \in \mathcal{C}$ , any  $x \in \Omega \cap \sigma$  of sedentarity  $I$  and all tangent vectors  $v_1, \dots, v_p, w_1, \dots, w_q \in \mathbb{L}(\sigma, x)$  we have

$$\langle \alpha_I(x); v_1, \dots, v_p, w_1, \dots, w_q \rangle = \langle \alpha'_I(x); v_1, \dots, v_p, w_1, \dots, w_q \rangle.$$

Let  $\mathcal{A}^{p,q}(\Omega)$  denote the set of equivalence classes of pairs  $(U, \alpha)$  as above.

**Remark 2.11.** By definition we have that  $\alpha$  and  $\alpha'$  define the same superform on  $\Omega$  if and only if for all  $I \subset [r]$  the superforms  $\alpha_I$  and  $\alpha'_I$  define the same superform on  $\Omega_I$ . Moreover, to determine if two superforms are equivalent when restricted to  $\Omega$ , it is enough to consider only points in the relative interior of facets. Furthermore,  $\mathcal{A}^{p,q}(\Omega)$  is independent of the underlying polyhedral complex  $\mathcal{C}$  and only depends on  $\Omega \subset \mathbb{T}^r$ .

The differential map  $d''$  and the wedge product both descend to forms in  $\mathcal{A}^{p,q}(\Omega)$  in the sense that if superforms  $\alpha, \beta \in \mathcal{A}^{p,q}(\Omega)$  are given by  $\alpha' \in \mathcal{A}^{p,q}(U)$  and  $\beta' \in \mathcal{A}^{p,q}(U')$  then defining  $d''\alpha$  to be given by  $d''\alpha'$  and  $\alpha \wedge \beta$  to be given by  $\alpha'|_{U \cap U'} \wedge \beta'|_{U \cap U'}$  is independent of the choices of  $\alpha'$  and  $\beta'$ .

For a polyhedral subspace  $X$  in  $\mathbb{T}^r$ , the functor on open subsets of  $X$  given by  $\Omega \mapsto \mathcal{A}_X^{p,q}(\Omega)$  will be denoted by  $\mathcal{A}_X^{p,q}$  or simply  $\mathcal{A}^{p,q}$  if the space of definition is clear. The next lemma shows that this is an acyclic sheaf, where by acyclicity we always mean with respect to both the functor of global sections and the functor of global sections with compact support.

**Lemma 2.12.** *For a polyhedral subspace  $X$  in  $\mathbb{T}^r$ , the presheaf*

$$\Omega \mapsto \mathcal{A}_X^{p,q}(\Omega)$$

*is a sheaf on  $X$ . Furthermore, this sheaf is fine, hence soft and acyclic.*

*Proof.* We start with the case  $X = \mathbb{T}^r$ . In this case, all of the sheaf properties are obvious except the gluing property. Given a collection of superforms agreeing on intersections, we can glue on each  $\mathbb{R}_I^r$  getting a collection of superforms  $\alpha_I$ , one on each  $\mathbb{R}_I^r$ . The condition of compatibility along the boundary strata is respected for the glued superforms since it is local and was respected for the superforms before gluing.

For the general case we rely on the existence of partitions of unity. Let  $(\Omega^l)$  be a collection of open sets and suppose that we have superforms  $\alpha^l \in \mathcal{A}^{p,q}(\Omega^l)$  which agree on the intersections of the  $\Omega^l$ 's and are the restrictions to  $X$  of superforms  $\beta^l \in \mathcal{A}^{p,q}(U^l)$ . We take a partition of unity  $(f^k)_{k \in K}$  subordinate to the cover  $(U^l)_{l \in L}$ . By definition there is a map  $s : K \rightarrow L$ , so that if  $s(k) = l$ , then  $f^k$  is supported on  $U_l$ . Thus  $\beta = \sum_{l \in L} \sum_{k: s(k)=l} f^k \beta^l$  defines

a superform on the union  $\cup_l \Omega^l$ . Moreover for a fixed  $l_0$  we have

$$\begin{aligned} \beta|_{\Omega^{l_0}} &= \sum_{l \in L} \sum_{k: s(k)=l} f^k|_{\Omega^{l_0}} \beta^l|_{\Omega^{l_0}} = \sum_{l \in L} \sum_{k: s(k)=l} f^k|_{\Omega^{l_0}} \alpha^l|_{\Omega^{l_0}} \\ &= \sum_{l \in L} \sum_{k: s(k)=l} f^k|_{\Omega^{l_0}} \alpha^{l_0} = \left( \sum_{k \in K} f^k|_{\Omega^{l_0}} \right) \alpha^{l_0} = \alpha^{l_0}. \end{aligned}$$

Therefore the superform given by  $\beta$  restricted to  $\cup_l \Omega^l$  gives the gluing of the superforms  $\alpha^l$  above. The fact that  $\mathcal{A}^{0,0}$  is fine follows from Lemma 2.7 and the sheaves  $\mathcal{A}^{p,q}$  are also fine since they are  $\mathcal{A}^{0,0}$ -modules via the wedge product. Softness and acyclicity for global sections follows from [Wel80, Chapter II, Proposition 3.5 & Theorem 3.11] respectively and acyclicity for sections with compact support follows from [Ive86, III, Theorem 2.7].  $\square$

**Definition 2.13.** The *support* of a superform  $\alpha$  is its support in the sense of sheaves, thus it consists of the points  $x$  which do not have a neighborhood  $\Omega_x$  such that  $\alpha|_{\Omega_x} = 0$ . We also have  $\text{supp}(\alpha) = \cup_I \text{supp}(\alpha_I)$ . The space of  $(p, q)$ -superforms with compact support on  $U$  is denoted  $\mathcal{A}_c^{p,q}(U)$ .

**2.2. Polyhedral spaces.** This subsection defines superforms on polyhedral spaces. These are spaces equipped with an atlas of charts to polyhedral subspaces in  $\mathbb{T}^r$ , with coordinate changes given by extensions of affine maps (cf. Definition 2.18). First we establish pullbacks of superforms along extended affine maps, which permit the gluing of the sheaves  $\mathcal{A}^{p,q}$  defined in the last subsection.

Let  $F : \mathbb{R}^{r'} \rightarrow \mathbb{R}^r$  be an affine map and let  $M_F$  denote the matrix representing the linear part of  $F$ . Let  $I$  be the set of  $i \in [r']$  such that the  $i$ -th column of  $M_F$  has only non-negative entries. Then  $F$  can be extended to a map

$$F : \left( \bigcup_{J \subset I} \mathbb{R}_J^{r'} \right) \rightarrow \mathbb{T}^r$$

by continuity, (equivalently, using the usual  $-\infty$ -conventions for arithmetic). The extended map is also denoted by  $F$ .

**Definition 2.14.** Let  $U' \subset \mathbb{T}^{r'}$  be an open subset, then a map  $F : U' \rightarrow \mathbb{T}^r$ , which is the restriction to  $U'$  of a map arising as above is called an *extended affine map*. Note that this only makes sense once we have  $\text{sed}(x) \subset I$  for all  $x \in U'$ . Similarly, for a polyhedral subspace  $X'$  and an open subset  $\Omega'$  of  $X'$  an extended affine map  $F : \Omega' \rightarrow \mathbb{T}^r$  is given by the restriction of an extended affine map to  $\Omega'$ . An extended affine map is called an *integral extended affine map*, if it is the extension of an integer affine map  $\mathbb{R}^{r'} \rightarrow \mathbb{R}^r$ , i.e. its linear part is induced by a map of the standard lattices  $\mathbb{Z}^{r'} \rightarrow \mathbb{Z}^r$ .

**Definition 2.15** (Pullback). Let  $U' \subset \mathbb{T}^{r'}$  be an open subset and  $F : U' \rightarrow \mathbb{T}^r$  be an extended affine map. Let  $U \subset \mathbb{T}^r$  be an open subset such that  $F(U') \subset U$ . Define

$$F : \{\text{sedentarities of points in } U'\} \rightarrow 2^{[r]}$$

$$I' \mapsto \text{sed}(F(x)) \text{ for some and then every } x \in \mathbb{R}_{I'}^{r'}.$$

Notice that this map respects inclusions. Then  $F$  induces an affine map  $F_{I'} : \mathbb{R}_{I'}^{r'} \rightarrow \mathbb{R}_{F(I')}^r$  with  $F_{I'}(U'_{I'}) \subset U_{F(I')}$ . The *pullback of the superform*  $\alpha = (\alpha_I)_I$  *along*  $F$  is the collection of superforms  $F^*(\alpha) := (F_{I'}^*(\alpha_{F(I')}))_{I'}$ . The next lemma shows that this defines a superform on  $U'$  and thus we have a pullback map  $F^* : \mathcal{A}^{p,q}(U) \rightarrow \mathcal{A}^{p,q}(U')$ .

**Lemma 2.16.** *The pullback of a  $(p, q)$ -superform  $\alpha$  on  $U \subset \mathbb{T}^r$  along an extended affine map  $F : U' \rightarrow U$  is a  $(p, q)$ -superform on  $U' \subset \mathbb{T}^{r'}$ .*

*Proof.* We have to verify the condition of compatibility of superforms along the strata. For  $J' \subset I'$  we have  $F(J') \subset F(I')$  and  $F_{I'} \circ \pi_{I', J'} = \pi_{F(I'), F(J')} \circ F_{J'}$ . Thus if  $\alpha \in \mathcal{A}^{p,q}(U)$  is determined by  $\alpha_{F(I')}$  on  $U_x$ , then we have

$$\begin{aligned} \pi_{I', J'}^*(F^*(\alpha)_{I'}) &= \pi_{I', J'}^* F_{I'}^*(\alpha_{F(I')}) \\ &= F_{J'}^*(\pi_{F(I'), F(J')}^*(\alpha_{F(I')})) \\ &= F_{J'}^*(\alpha_{F(J')}) \\ &= (F^*(\alpha))_{J'}, \end{aligned}$$

which shows that  $F^*(\alpha)$  is determined by  $F^*(\alpha)_{I'}$  on  $F^{-1}(U_x)$ . This shows the required compatibility.  $\square$

**Lemma 2.17.** *Let  $X \subset \mathbb{T}^r$  and  $X' \subset \mathbb{T}^{r'}$  be polyhedral subspaces and let  $\Omega \subset X$  and  $\Omega' \subset X'$  be open subsets. If  $F : \Omega' \rightarrow \Omega$  is an extended affine map, then there exists a well defined pullback  $F^* : \mathcal{A}^{p,q}(\Omega) \rightarrow \mathcal{A}^{p,q}(\Omega')$ , which is induced by the pullback in Definition 2.15. Moreover, the pullback is functorial and commutes with the differential  $d''$  and the wedge product.*

*Proof.* Let  $\alpha \in \mathcal{A}^{p,q}(\Omega)$ , then there exist open subsets  $U' \subset \mathbb{T}^{r'}$  and  $U \subset \mathbb{T}^r$  such that  $\alpha$  is defined by some  $\beta \in \mathcal{A}^{p,q}(U)$ ,  $F(U') \subset U$  and  $U' \cap X' = \Omega'$ . Now the pullback  $F^*(\beta) \in \mathcal{A}^{p,q}(U')$  defines a superform on  $\Omega'$ . Set this to be  $F^*(\alpha)$ . To see that this is independent of the choice of  $\beta$  we suppose that  $\gamma$  is another superform on an open set defining  $\alpha$  on  $\Omega$ . After intersecting their respective domains of definition, we may assume that  $\beta$  and  $\gamma$  are defined on the same open set  $U$ . Since  $\beta|_{\Omega} = \gamma|_{\Omega}$  we have that  $\beta|_{\Omega_{F(I')}} = \gamma|_{\Omega'_{F(I'')}}$  for all  $I' \subset [r']$ . Since the pullback via affine maps between vector spaces is well defined on polyhedral complexes (cf. [Gub13, 3.2]), we have  $F_{I'}^*(\beta)|_{\Omega'_{I'}} = F_{I'}^*(\gamma)|_{\Omega'_{I'}}$  for all  $I' \subset [r']$  and therefore  $F^*(\beta)|_{\Omega'} = F^*(\gamma)|_{\Omega'}$ , so that the pullback is well defined. The last two statements are direct consequences of the definition of pullbacks of forms along extended affine maps and the fact that pullback by affine maps is functorial and commutes with  $d''$  and the wedge product.  $\square$

We can now consider spaces equipped with an atlas of charts to polyhedral subspaces in  $\mathbb{T}^r$ . The following definition is a generalization of the definition of tropical spaces given in, for example, [Mik06], [MZ13], [BIMS15]. We do not require our polyhedral subspaces to be rational, also the transition maps are required only to be extended affine maps, not integral affine. We also remove the finite type condition on the charts, (cf. [MZ13, Definition 1.2]).

**Definition 2.18.** A *polyhedral space*  $X$  is a paracompact, second countable Hausdorff topological space with an atlas of charts  $A = (\varphi_i : U_i \rightarrow \Omega_i \subset X_i)_{i \in I}$  such that:

- i) The  $U_i$  are open subsets of  $X$ , the  $\Omega_i$  are open subsets of  $X_i$ , which are polyhedral subspaces in some  $\mathbb{T}^{r_i}$ , and  $\varphi_i : U_i \rightarrow \Omega_i$  is a homeomorphism for all  $i$ ;



ii) For all  $i, j \in I$  the transition map

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \rightarrow X_i$$

is an extended affine map.

A polyhedral space  $X$  is *regular at infinity* if  $X_i = Y_i \times \mathbb{T}^{t_i} \subset \mathbb{R}^{r_i} \times \mathbb{T}^{t_i}$ , where  $Y_i$  is a polyhedral subspace of  $\mathbb{R}^{r_i}$  for every chart  $\varphi_i$ .

As in usual manifold theory two atlases on  $X$  are considered equivalent if their common refinement is an atlas on  $X$ . Given a polyhedral space  $X$ , there exists a maximally saturated atlas. We implicitly work with this atlas even if  $X$  is defined with a more manageable collection of charts.

The *dimension* of  $X$  is the maximal dimension among polyhedra which intersect the  $\Omega_i$ . The polyhedral complex is *pure dimensional* if the dimension of the maximal, with respect to inclusion, polyhedra intersecting the open sets  $\Omega_i \subset X_i$  is constant.

**Definition 2.19.** Let  $X$  be a polyhedral space with atlas  $A = (\varphi_i : U_i \rightarrow \Omega_i \subset X_i)_{i \in I}$ . Define the sheaf  $\mathcal{A}_X^{p,q}$  of  $(p, q)$ -superforms on  $X$  to be the gluing of the sheaves  $\mathcal{A}_{U_i}^{p,q}$ , which is the pullback of the sheaves  $\mathcal{A}_{\Omega_i}^{p,q}$  via  $\varphi_i$ . The pullback of forms along the charts  $\varphi_i$  is well defined and functorial, so this gives a well defined sheaf of superforms on  $X$ . We also again denote the sections with compact support by  $\mathcal{A}_c^{p,q}(X)$ .

The next proposition is immediate following the local considerations in Section 2.1.

**Proposition 2.20.** Let  $X$  be a polyhedral space, then the differential  $d''$  and the wedge product of superforms on  $X$  are well defined and for each  $p \in \mathbb{N}$  there is a complex

$$0 \rightarrow \mathcal{A}_X^{p,0} \xrightarrow{d''} \mathcal{A}_X^{p,1} \rightarrow \dots$$

If  $X$  is  $n$ -dimensional, then  $\mathcal{A}_X^{p,q} = 0$  for  $\max(p, q) > n$ . The sheaves  $\mathcal{A}_X^{p,q}$  are fine, hence soft and acyclic.

**Definition 2.21.** Let  $X$  be a polyhedral space, then the *Dolbeault cohomology* of superforms is defined as  $H_{d''}^{p,q}(X) := H^q(\mathcal{A}_X^{p,\bullet}(X), d'')$  and the *Dolbeault cohomology of superforms with compact support* is defined as  $H_{d'',c}^{p,q}(X) := H^q(\mathcal{A}_{X,c}^{p,\bullet}(X), d'')$ .

**Example 2.22.** Let  $[0, 1]$  be the closed unit interval and define the following charts:

$$\begin{aligned} \varphi_0 : [0, 1] &\cong \mathbb{T}^1 & \varphi_1 : (0, 1] &\cong \mathbb{T}^1 \\ x &\mapsto \tan((x - 1/2)\pi) & x &\mapsto -\tan((x - 1/2)\pi). \end{aligned}$$

The interval  $[0, 1]$  equipped with these two charts defines a polyhedral space, denote it by  $X$ . The single transition map for this atlas is  $\varphi_0 \circ \varphi_1^{-1} : \mathbb{R} \rightarrow \mathbb{R} = (x \mapsto -x)$ . In Example 2.6, we saw that  $(0, 0)$ -superforms on  $\mathbb{T}$  are functions which are locally constant around  $-\infty$ . Thus  $(0, 0)$ -superforms on  $X$  are locally constant around both 0 and 1. Furthermore, similar to Example 2.6, superforms of positive degree vanish locally at the two boundary points of  $X$ .

The space  $[0, 1]$  can also be equipped with an atlas consisting of a single chart which is just the inclusion  $[0, 1] \hookrightarrow \mathbb{R}$ . Denote this polyhedral space by  $\tilde{X}$ . Then superforms in  $\mathcal{A}^{0,0}(\tilde{X})$  are just smooth functions on  $[0, 1]$  in the usual sense, since superforms are not required to satisfy any compatibility conditions. Also the superforms  $d'x$ ,  $d''x$  and  $d'x \wedge d''x$  are nowhere vanishing superforms of positive degree.

### 3. COMPARISON OF COHOMOLOGIES

In this section we show that the Dolbeault cohomology of superforms agrees with tropical cohomology on polyhedral spaces. Subsection 3.1 recalls the definition of tropical cohomology, and also defines basic open subsets of polyhedral spaces. We then recall the description

of tropical cohomology in terms of sheaves. In Subsection 3.2 we show that Dolbeault cohomology of superforms is also equivalent to the cohomology of certain sheaves. We then calculate the sections of these sheaves over basic open subsets and deduce from this that the sheaves defining tropical and Dolbeault cohomologies agree.

**3.1. Tropical cohomology.** This subsection describes tropical cohomology from [IKMZ], see also [MZ13], [BIMS15].

**Definition 3.1.** Let  $\mathcal{C}$  be a polyhedral complex in  $\mathbb{T}^r$ . For  $\sigma \in \mathcal{C}$ , there is an  $I \subset [r]$  such that  $\text{int}(\sigma) \subset \mathbb{R}_I^r$ . The  $p$ -th *multitangent* and *multicotangent space* of  $\mathcal{C}$  at  $\sigma$  are the vector subspaces respectively

$$\mathbf{F}_p(\sigma) = \sum_{\tau \in \mathcal{C}_I: \sigma \prec \tau} \Lambda^p \mathbb{L}(\tau) \subset \Lambda^p \mathbb{R}_I^r \quad \text{and} \quad \mathbf{F}^p(\sigma) = \left( \sum_{\tau \in \mathcal{C}_I: \sigma \prec \tau} \Lambda^p \mathbb{L}(\tau) \right)^*.$$

For  $\sigma \prec \tau$  for every  $p$  there is a map  $i_{\tau\sigma} : \mathbf{F}_p(\tau) \rightarrow \mathbf{F}_p(\sigma)$ , which is an inclusion of vector spaces if  $\sigma$  and  $\tau$  are of the same sedentarity. Otherwise, if  $\sigma$  is of sedentarity  $J$  and  $\tau$  of sedentarity  $I$ , the map  $i_{\tau\sigma}$  is given by the composition of the projection  $\pi_{IJ}$  and the above inclusion. On the dual spaces  $\mathbf{F}^p(\sigma)$  the maps are of course reversed  $r_{\tau\sigma} : \mathbf{F}^p(\sigma) \rightarrow \mathbf{F}^p(\tau)$ .

Notice also that a polyhedral space  $X$  comes equipped with a stratification by the dimension of cells in the coarsest polyhedral structure of each  $\Omega_i$ , (cf. [MZ13, Section 1.5] for more details). This is known as the combinatorial stratification. The definition of the multitangent spaces above can be extended to any locally polyhedral space by way of the charts. For any combinatorial stratum  $\sigma$  of  $X$ , the vector space  $\mathbf{F}^p(\sigma)$  is well defined, since the coordinate changes are restrictions of extended affine maps. However, to define the maps  $r_{\tau\sigma} : \mathbf{F}^p(\sigma) \rightarrow \mathbf{F}^p(\tau)$  for a polyhedral space  $X$  we must impose an additional condition, which we call here a *face structure* on  $X$  (cf. Definition 3.2). This condition also appeared in [MZ13, Definition 1.10].

**Definition 3.2.** Let  $X$  be a polyhedral space with atlas  $A = (\varphi_i : U_i \rightarrow \Omega_i \subset X_i)_{i \in I}$ . A *face structure* on  $X$ , consists of fixed polyhedral structures  $\mathcal{C}_i$  on  $X_i$  for each  $i$  and a finite number of closed sets  $\{\bar{\sigma}_k\}$ , called *facets*, which cover  $X$ , such that

- i) each facet  $\bar{\sigma}_k$  is contained in some chart  $U_i$  for some  $i$  such that  $\varphi_i(\bar{\sigma}_k)$  is the intersection of  $\Omega_i$  with a facet  $\tau_{ik}$  of the polyhedral complex  $\mathcal{C}_i$ ;
- ii) for any collection of facets  $\mathcal{S}$  and  $\bar{\sigma}_l \in \mathcal{S}$  the image of the intersection  $\cap_{\bar{\sigma}_k \in \mathcal{S}} \bar{\sigma}_k$  in the chart  $\varphi_i : U_i \rightarrow X_i$  containing  $\bar{\sigma}_l$  is the intersection of  $\Omega_i$  with a face of  $\tau_{il}$ .

Given a face structure of  $X$  a *face* of  $X$  is an intersection of facets. A *face of a facet*  $\sigma$  is the intersection of  $\sigma$  with a face of  $X$ .

Note that every open subset of the support of a polyhedral subcomplex of  $\mathbb{T}^r$  is a polyhedral space with a face structure. For example, one can take the facets to be the intersections of (maximal) polyhedra of the polyhedral complex with the open subset. Notice also that the combinatorial stratification of a polyhedral space does not necessarily have all of the properties of a face structure.

Two face structures on  $X$  are *equivalent* if there exists a common refinement, i.e. a face structure  $\{\bar{\sigma}_k\}$  on  $X$  such that every facet  $\bar{\sigma}_k$  is contained in a facet of each of the two original face structures.

A face structure on  $X$  provides canonical maps  $i_{\tau\sigma}$ ,  $r_{\tau\sigma}$  between the multitangent and multicotangent spaces respectively, for faces  $\sigma \prec \tau$  of  $X$ . These maps are induced by the maps between the multitangent and multicotangent spaces of the images of the faces under a chart of the polyhedral space.

**Example 3.3.** We again consider the polyhedral space  $X$ , given by equipping the space  $[0, 1]$  with two charts to  $\mathbb{T}^1$  as in Example 2.22. Notice that  $[0, 1]$  cannot be the only facet of a face structure on  $X$  since it is not contained in a single chart. Choosing, for both

charts, the polyhedral structure on  $\mathbb{T}$  with facets  $[-\infty, 0]$  and  $[0, \infty)$  and on  $X$  taking facets  $\bar{\sigma}_1 = [0, 1/2]$  and  $\bar{\sigma}_2 = [1/2, 1]$  gives a face structure.

Denote by  $\Delta_q$  the standard  $q$ -simplex in  $\mathbb{R}^{q+1}$ .

**Definition 3.4.** Let  $X$  be a polyhedral space together with a face structure  $\mathcal{C}$  on  $X$ .

- i) For every facet  $\tau \in \mathcal{C}$ , we write  $C_q(\tau)$  for the free real vector space generated by continuous maps  $\delta : \Delta_q \rightarrow \tau$  such that the image of the relative interior of  $\Delta_q$  is contained in the relative interior of  $\tau$  and in addition the image of each open face  $\Delta'_q \subset \Delta_q$  is contained in the relative interior of a face of  $\tau$ . The space of *tropical*  $(p, q)$ -cells on  $X$  with respect to  $\mathcal{C}$  is

$$C_{p,q}(X) := \bigoplus_{\tau \in \mathcal{C}} \mathbf{F}_p(\tau) \otimes C_q(\tau).$$

- ii) For  $\delta \in C_q(\tau)$  write  $\partial\delta = \sum_{k=0}^q (-1)^{\epsilon_k} \delta^k$  for the usual boundary map, considered as a map  $C_q(\tau) \rightarrow \bigoplus_{\sigma \prec \tau} C_{q-1}(\sigma)$ . For every  $\sigma \prec \tau$  in  $\mathcal{C}$  we have the map of multitangent spaces,  $i_{\tau\sigma} : \mathbf{F}_p(\tau) \rightarrow \mathbf{F}_p(\sigma)$ . For  $v \otimes \delta \in \mathbf{F}_p(\tau) \otimes C_q(\tau)$  we define the *boundary operator* by

$$\partial(v \otimes \delta) := \sum_{k=0}^q (-1)^{\epsilon_k} v^k \otimes \delta^k \in C_{p,q-1}(X),$$

where  $v^k := i_{\tau\sigma}(v)$  when  $\delta^k(\Delta_{q-1}) \subset \text{relint}(\sigma)$ . We obtain complexes  $(C_{p,\bullet}(X), \partial)$  of real vector spaces.

- iii) We define *tropical homology groups* by

$$H_{p,q}^{\text{trop}}(X) := H_q(C_{p,\bullet}(X), \partial).$$

Dually, we define *tropical cochains* by  $C^{p,q}(X) := \text{Hom}(C_{p,q}(X), \mathbb{R})$  and the *tropical cohomology* of  $X$  as the cohomology of the dual complex

$$H_{\text{trop}}^{p,q}(X) := H^q(C^{p,\bullet}(X), \partial^*).$$

- iv) We say that  $\alpha \in C^{p,q}(X)$  has *compact support* if there exists a compact subset  $K_\alpha \subset X$  such that  $\alpha(v \otimes \delta) \neq 0$  implies  $\delta(\Delta_q) \cap K_\alpha \neq \emptyset$ . The cochains with compact support form a complex  $C_c^{p,\bullet}(X)$  and we define *tropical cohomology with compact support* by

$$H_{\text{trop},c}^{p,q}(X) := H^q(C_c^{p,\bullet}(X), \partial^*).$$

**Remark 3.5.** There are also cellular versions of tropical homology and cohomology, cf. [MZ13, Section 2.2]. The advantage of the cellular versions is that they are given by finitely generated complexes. Here, we use the sheaf theoretic approach to tropical cohomology which was shown to be equivalent to the singular definition above in [MZ13, Theorem 2.8]. We will recall the definition of this sheaf of vector spaces and give a sketch of the argument which proves the equivalence of tropical cohomology and the cohomology of this sheaf. The purpose of outlining this argument is to show that it also applies for compactly supported cohomology.

Let  $\mathcal{C}$  be a polyhedral complex in  $\mathbb{T}^r$ . From the vector spaces  $\mathbf{F}^p(\sigma)$ , it is possible to construct a sheaf on  $|\mathcal{C}| \subset \mathbb{T}^r$  following the lines of [MZ13, Section 2.3]: For each open set  $\Omega \subset |\mathcal{C}|$ , consider the poset  $P(\Omega)$ , whose elements are the connected components  $\sigma$  of faces of  $\mathcal{C}$  intersecting with  $\Omega$ . The elements of  $P(\Omega)$  are ordered by inclusion and if  $\sigma \prec \tau$  recall there are maps  $r_{\tau\sigma} : \mathbf{F}^p(\sigma) \rightarrow \mathbf{F}^p(\tau)$ .

**Definition 3.6.** For an open set  $\Omega \subset |\mathcal{C}|$  define the vector space

$$\mathcal{F}^p(\Omega) := \varprojlim_{\sigma \in P(\Omega)} \mathbf{F}^p(\sigma).$$

As outlined in [MZ13] the above defines a constructible sheaf of vector spaces on  $|\mathcal{C}|$  in  $\mathbb{T}^r$ . These sheaves do not depend on the polyhedral structure  $\mathcal{C}$  and thus are well defined for polyhedral subspaces. For a polyhedral space  $X$ , the sheaves  $\mathcal{F}_X^p$  are defined by gluing along charts. Note that this definition does not require a face structure on  $X$ . The sheaves  $\mathcal{F}_X^p$  can also be defined by using the combinatorial stratification rather than a polyhedral structure in each chart, as done in [MZ13].

**Definition 3.7.** A subset  $\Delta \subset \mathbb{T}^r$  is an *open cube* if it is a product of intervals which are either  $(a_i, b_i)$  or  $[-\infty, c_i)$  for  $a_i \in \mathbb{T}$ ,  $b_i, c_i \in \mathbb{R} \cup \{\infty\}$ .

For a polyhedral complex  $\mathcal{C}$  in  $\mathbb{T}^r$ , a subset  $\Omega$  of  $|\mathcal{C}|$  is called a *basic open* subset if there exists an open cube  $\Delta \subset \mathbb{T}^r$  such that  $\Omega = |\mathcal{C}| \cap \Delta$  and such that the set of polyhedra of  $\mathcal{C}$  intersecting  $\Omega$  has a unique minimal element. Note that the sedentarity of the minimal polyhedron of  $\Omega$  is the maximal sedentarity among points in  $\Omega$ .

Let  $X$  be a polyhedral space with atlas  $(\varphi_i : U_i \rightarrow \Omega_i \subset X_i)$ , such that for each  $i$  we have a fixed polyhedral structure  $\mathcal{C}_i$  on  $X_i$ . Then we say that an open subset  $U$  is a *basic open* subset (with respect to these structures) if there exists a chart  $\varphi : U_i \rightarrow X_i$  such that  $U \subset U_i$  and  $\varphi(U)$  is a basic open subset of  $|\mathcal{C}_i|$ .

**Lemma 3.8.** *Let  $\mathcal{C}$  be a polyhedral complex in  $\mathbb{T}^r$ , then the basic open sets form a basis of the topology on  $|\mathcal{C}|$ . Further, if  $\Omega$  is a basic open subset of a polyhedral complex  $|\mathcal{C}|$  of sedentarity  $I$ , then  $\Omega_I$  is a basic open subset of the polyhedral complex  $|\mathcal{C}_I|$  in  $\mathbb{R}_I^r$ .*

*Proof.* Basic open sets form a basis of the topology of  $|\mathcal{C}|$  since open cubes form a basis of the topology of  $\mathbb{T}^r$ . For the second statement, we have that  $\Omega_I = |\mathcal{C}_I| \cap \Delta_I$  and the minimal polyhedron of  $\Omega_I$  is the same as the one of  $\Omega$ , so the lemma is proven.  $\square$

**Lemma 3.9.** *Let  $\mathcal{C}$  be a polyhedral complex in  $\mathbb{T}^r$  and  $\Omega$  a basic open subset of  $|\mathcal{C}|$ . Then*

$$\mathcal{F}^p(\Omega) = \mathbf{F}^p(\sigma),$$

*where  $\sigma$  is the minimal polyhedron of  $\mathcal{C}$ .*

*Proof.* Let  $\Delta$  be an open cube such that  $\Omega = \Delta \cap |\mathcal{C}|$  and suppose that  $I$  is such that  $\text{int}(\sigma) \subset \mathbb{R}_I^r$ . Then  $\Omega \cap \sigma = \Delta \cap \sigma = (\Delta \cap \mathbb{R}_I^r) \cap \text{int}(\sigma)$  is connected, since it is the intersection of two convex sets. Thus the poset  $P(\Omega)$  has  $\Omega \cap \sigma$  as its unique minimal element and the lemma follows.  $\square$

Next we compute  $H_{\text{trop}}^{p,q}(\Omega)$  for a basic open set  $\Omega$ .

**Proposition 3.10.** *Let  $\Omega$  be a basic open subset of a polyhedral subspace  $|\mathcal{C}|$ , for a polyhedral complex  $\mathcal{C}$  in  $\mathbb{T}^r$ . Suppose further that  $|\mathcal{C}|$  is regular at infinity. Then*

$$H_{\text{trop}}^{p,q}(\Omega) = 0$$

*for  $q > 0$ . Furthermore, we have canonical isomorphisms*

$$H_{\text{trop}}^{p,0}(\Omega) = \mathbf{F}^p(\sigma)$$

*where  $\sigma$  is the minimal polyhedron of  $\Omega$ .*

*Proof.* Suppose first that the minimal polyhedron of  $\Omega$  is of sedentarity  $\emptyset$  and denote it by  $\sigma$ . Choose a point  $x_0 \in \Omega$  which is in the relative interior of  $\sigma$ . By performing a translation of  $\Omega$  we may assume that  $x_0 = 0$ .

Consider the deformation retraction of  $\mathbb{R}^r$  to the origin,  $f : \mathbb{R}^r \times [0, 1] \rightarrow \mathbb{R}^r$  given by  $f(x, t) = (1 - t)x$ . Then  $f$  restricted to  $\Omega$  is again a deformation retract, moreover it preserves the faces of  $\Omega$ . Let  $f_1(x) = f(x, 1) = 0$ , then for all  $p$  there is a map of chain complexes

$$f_1^* : C^\bullet(x_0; \mathbf{F}^p(\sigma)) \rightarrow C_{\text{trop}}^{p,\bullet}(X).$$

The map  $f_1^*$  can be seen to be homotopic to the identity map, following the usual arguments for constant coefficients. The standard argument extends to this situation since the homotopy  $f$  respects the face structure of  $\Omega$ .

Since  $x_0$  is a point we have  $H^q(x_0; \mathbf{F}^p(\sigma)) = 0$  for  $q \neq 0$  and  $H^0(x_0; \mathbf{F}^p(\sigma)) = \mathbf{F}^p(\sigma)$ . Therefore, the proposition is proven when  $\Omega$  has minimal polyhedron of sedentarity  $\emptyset$ .

Now suppose that  $\Omega$  is a basic open subset of  $|\mathcal{C}| \subset \mathbb{T}^r$  with minimal face  $\sigma$  of sedentarity  $I$ . Since  $\Omega$  is a basic open subset and  $|\mathcal{C}|$  is regular at infinity,  $\Omega$  is an open subset of  $V \times \mathbb{T}^{|I|}$ , where  $V$  is a fan in  $\mathbb{R}^s$ . Firstly, there is a deformation retraction from  $V \times \mathbb{T}^{|I|}$  to  $V \times \{(-\infty, \dots, -\infty)\}$  given by

$$g(x, t) = (1 - t)x - \frac{1}{1 - t} \sum_{i=s+1}^{r+|I|} e_i \quad \text{where } x \in V \times \mathbb{T}^{|I|} \text{ and } t \in [0, 1].$$

Notice that  $g(x, 0) = \text{id}$  and denote  $g_1 = g(x, 1)$ . Since the homotopy  $g$  respects the faces of  $\Omega$ , for every  $p$  there is a map of chain complexes  $g_1^* : C_{\text{trop}}^{p, \bullet}(\Omega) \rightarrow C_{\text{trop}}^{p, \bullet}(\Omega_I)$ .

It is once again the case that  $g_1^*$  is homotopic to the identity, so that there are isomorphisms  $H_{\text{trop}}^{p, q}(\Omega) \cong H_{\text{trop}}^{p, q}(\Omega_I)$ . Now  $\Omega_I \subset \mathbb{R}^r$  and we can apply the above argument for basic open subsets with minimal face of sedentarity  $\emptyset$  to conclude that  $H_{\text{trop}}^{p, q}(\Omega) = 0$  if  $q \neq 0$  and  $H_{\text{trop}}^{p, q}(\Omega) \cong \mathbf{F}^p(\sigma)$  for  $q = 0$ . This proves the proposition.  $\square$

**Proposition 3.11.** *For a polyhedral space  $X$  that is regular at infinity and has a face structure there are canonical isomorphisms*

$$H_{\text{trop}}^{p, q}(X) \cong H^q(X, \mathcal{F}^p) \text{ and } H_{\text{trop}, c}^{p, q}(X) \cong H_c^q(X, \mathcal{F}^p).$$

*Proof.* The first isomorphism is proven in [MZ13, Proposition 2.8]. It follows from standard arguments in algebraic topology, which are for example laid out in [Ram05, p. 110-113] in the classical setting of singular cohomology and cohomology of constant sheaves. The standard arguments adapt to the situation of non-constant coefficients by defining the sheaf  $\underline{C}^{p, q}$  as the sheaf associated to the presheaf  $\Omega \mapsto C^{p, q}(\Omega)$ . Now the complex  $(\underline{C}^{p, \bullet}(\Omega), \partial^*)$  is quasi-isomorphic to  $(C^{p, \bullet}(\Omega), \partial^*)$ . The proof works in the standard way, since barycentric subdivisions preserve the coefficients. Thus by Lemma 3.9 and Proposition 3.10 the complex  $(\underline{C}^{p, \bullet}, \partial^*)$  is a resolution of  $\mathcal{F}^p$ . The sheaves  $\underline{C}^{p, q}$  are flasque because the presheaves are flasque and satisfy the glueing axiom, thus the map into the sheafification is surjective. Therefore  $(\underline{C}^{p, \bullet}(X), \partial^*)$  and the sections with compact support,  $(\underline{C}_c^{p, \bullet}(X), \partial^*)$ , calculate  $H^q(X, \mathcal{F}^p)$  and  $H_c^q(X, \mathcal{F}^p)$  respectively. These complexes are once again quasi-isomorphic to the chain complexes of tropical  $(p, \bullet)$ -chains, respectively tropical  $(p, \bullet)$ -chains with compact support, from Definition 3.4. This concludes the proposition.  $\square$

**3.2. Dolbeault cohomology of superforms.** In this subsection we prove a local exactness result for superforms, called the Poincaré lemma (cf. Theorem 3.12). Using the acyclicity established in the last section, we identify the Dolbeault cohomology of superforms with the cohomology of certain sheaves. We then calculate the sections of these sheaves on basic open sets in Proposition 3.16. The next theorem is a straightforward extension of [Jel15, Theorem 2.16].

**Theorem 3.12** (Poincaré lemma). *Let  $X$  be a polyhedral space and  $U \subset X$  an open subset. Let  $\alpha \in \mathcal{A}^{p, q}(U)$  with  $q > 0$  and  $d''\alpha = 0$ . Then for every  $x \in U$  there exists an open subset  $V \subset X$  with  $x \in V$  and a superform  $\beta \in \mathcal{A}^{p, q-1}(V)$  such that  $d''\beta = \alpha|_V$ .*

*Proof.* After shrinking  $U$ , we may assume that there is a chart  $\varphi : U \rightarrow \Omega$  for  $\Omega$  an open subset of the support of a polyhedral complex  $\mathcal{C}$  in  $\mathbb{T}^r$ . Since this question is purely local, we may prove the statement for an open subset  $\Omega \subset |\mathcal{C}|$  for a polyhedral complex  $\mathcal{C}$  in  $\mathbb{T}^r$ . If  $\text{sed}(x) = \emptyset$  then this is shown in [Jel15, Theorem 2.16].

For the general case, let  $I = \text{sed}(x)$  and after possibly shrinking  $\Omega$  we may assume that  $I$  is the unique maximal sedentarity among points in  $\Omega$  and  $\alpha$  is determined by  $\alpha_I$  on  $\Omega$ . After possibly shrinking  $\Omega$  again, by the case  $I = \emptyset$ , we have  $\beta_I \in \mathcal{A}^{p, q}(\Omega_I)$  such that  $d''\beta_I = \alpha_I$ . For each  $J \subset I$ , set  $\beta_J = \pi_{IJ}^* \beta_I$ , then this determines a superform  $\beta \in \mathcal{A}^{p, q}(\Omega)$  and since the affine pullback commutes with  $d''$ , we have  $d''\beta_J = \alpha_J$ , hence  $\beta$  has the required property and the theorem is proven.  $\square$

**Definition 3.13.** For  $X$  a polyhedral space and  $p \in \mathbb{N}$  we define the sheaf

$$\mathcal{L}_X^p := \ker(d'' : \mathcal{A}_X^{p,0} \rightarrow \mathcal{A}_X^{p,1}).$$

Again we omit the subscript  $X$  on  $\mathcal{L}_X^p$  if the space  $X$  is clear from context.

**Corollary 3.14.** For a polyhedral space  $X$  and all  $p \in \mathbb{N}$ , the complex

$$0 \rightarrow \mathcal{L}^p \rightarrow \mathcal{A}^{p,0} \xrightarrow{d''} \mathcal{A}^{p,1} \xrightarrow{d''} \mathcal{A}^{p,2} \rightarrow \dots$$

of sheaves on  $X$  is exact. Furthermore it is an acyclic resolution, we thus have canonical isomorphisms

$$H^q(X, \mathcal{L}^p) \cong H_{d''}^{p,q}(X) \quad \text{and} \quad H_c^q(X, \mathcal{L}^p) \cong H_{d'',c}^{p,q}(X).$$

*Proof.* Exactness is a direct consequence of Theorem 3.12 and Definition 3.13 and acyclicity comes from Proposition 2.20.  $\square$

**Example 3.15.** We calculate the dimensions of the Dolbeault cohomology for the polyhedral spaces from Example 2.22. Let  $h^{p,q}(X) := \dim H_{d''}^{p,q}(X)$  for all  $p, q$ .

It is easy to see that, for any polyhedral space,  $\mathcal{L}^0$  is the constant sheaf with stalk  $\mathbb{R}$ . By Corollary 3.14 and comparison with singular cohomology, we obtain  $h^{0,0}(X) = 1$  and  $h^{0,1}(X) = 0$ . This argument shows that, in general, the cohomology groups  $H^{0,q}$  do not depend on the atlas of  $X$ . For the polyhedral space  $X$  from Example 2.22, recall that the compatibility condition for superforms along the boundary strata implies that all smooth functions are locally constant at points  $0, 1 \in X$ . Also all superforms of positive degree have support away from the boundary points. Thus  $(1, 0)$  and  $(1, 1)$ -superforms on  $X$  are simply forms on  $\mathbb{R}$  with compact support. Fix a coordinate  $x$  on  $\mathbb{R}$ . Then  $\alpha \in \mathcal{A}_c^{1,0}(\mathbb{R})$  is of the form  $\alpha = f dx$  with  $f \in C_c^\infty(\mathbb{R})$  and is closed precisely if  $\frac{\partial f}{\partial x} = 0$ . This however just means  $f = 0$  and hence  $\alpha = 0$ , thus  $h^{1,0}(X) = 0$ . For  $h^{1,1}(X)$  note that a superform  $f dx \wedge d''x$  with  $f \in C_c^\infty(\mathbb{R})$  is exact precisely if  $f$  has an antiderivative with compact support in  $\mathbb{R}$ . This is the case precisely when  $\int_{\mathbb{R}} f = 0$ , thus  $h^{1,1}(X) = 1$ . Notice that  $h^{0,0}(X) = h^{1,1}(X)$  and  $h^{0,1}(X) = h^{1,0}(X)$ . Example 2.22 also considered the polyhedral space  $\tilde{X}$  given by  $[0, 1]$  with the inclusion  $[0, 1] \hookrightarrow \mathbb{R}$  as the only chart. The dimensions of the cohomology groups for  $\tilde{X}$  are

$$h^{0,0}(\tilde{X}) = 1, \quad h^{0,1}(\tilde{X}) = 0, \quad h^{1,0}(\tilde{X}) = 1 \quad \text{and} \quad h^{1,1}(\tilde{X}) = 0.$$

We will revisit this in Example 4.10.

**Proposition 3.16.** Let  $\mathcal{C}$  be a polyhedral complex in  $\mathbb{T}^r$  and  $\Omega$  be basic open set of  $|\mathcal{C}|$  with minimal polyhedron  $\sigma$  of sedentarity  $I$ . Then we have

$$\mathcal{L}^p(\Omega) = \left( \sum_{\tau \in \mathcal{C}_I: \sigma \prec \tau} \Lambda^p \mathbb{L}(\tau) \right)^*.$$

For basic open subsets  $\Omega' \subset \Omega$ , the restriction maps  $\mathcal{L}^p(\Omega) \rightarrow \mathcal{L}^p(\Omega')$  are given by the dual of the inclusion

$$\sum_{\tau \in \mathcal{C}_I: \sigma' \prec \tau} \Lambda^p \mathbb{L}(\tau) \hookrightarrow \sum_{\tau \in \mathcal{C}_I: \sigma \prec \tau} \Lambda^p \mathbb{L}(\tau)$$

when the minimal polyhedron  $\sigma'$  of  $\Omega'$  is also of sedentarity  $I$ . If the sedentarity of  $\sigma'$  is  $J \subsetneq I$  then the restriction map is dual to the map

$$\sum_{\tau \in \mathcal{C}_J: \sigma' \prec \tau} \Lambda^p \mathbb{L}(\tau) \rightarrow \sum_{\tau \in \mathcal{C}_I: \sigma \prec \tau} \Lambda^p \mathbb{L}(\tau)$$

which is the composition of projection  $\pi_{IJ}$  and the above inclusion.

*Proof.* We start with the case  $I = \emptyset$ , thus  $\Omega \subset \mathbb{R}^r$ . Given a  $(p, 0)$ -superform in the kernel of  $d''$ , the strategy is to construct a superform whose coefficient functions are all constant and to show that this superform agrees with the original superform on  $\Omega$ .

Recall that  $\sigma$  is the minimal polyhedron of the basic open set  $\Omega$ . Set  $V = \sum_{\sigma \prec \tau} \Lambda^p \mathbb{L}(\tau)$ . There is a natural map  $V^* \rightarrow \mathcal{L}^p(\Omega) \subset \mathcal{A}^{p,0}(\Omega)$  and this is clearly injective. To show surjectivity choose  $v_1, \dots, v_k$  such that each  $v_i \in \Lambda^p \mathbb{L}(\tau)$  for some  $\tau$  and  $v_1, \dots, v_k$  is a basis of  $V$  and extend to a basis  $v_1, \dots, v_k, v_{k+1}, \dots, v_s$  of  $\Lambda^p \mathbb{R}^r$ . Write  $\alpha \in \mathcal{L}^p(\Omega)$  as

$$\alpha = \sum_{i=1}^s f_i d' v_i$$

for  $f_i$  smooth functions on open subsets of  $\mathbb{R}^r$  and  $d' v_1, \dots, d' v_k$  is the dual to our fixed basis. By definition we therefore have  $f_i = \langle \alpha, v_i \rangle$ .

Notice that for any  $\sigma \prec \tau$ , the set  $\Omega \cap \tau$  is connected, since it is the intersection of an open cube and a polyhedron. For a fixed  $\tau$  such that  $\sigma \prec \tau$  and a fixed vector  $w_\tau \in \Lambda^p \mathbb{L}(\tau)$  define the function

$$(1) \quad \langle \alpha, w_\tau \rangle : \tau \cap \Omega \rightarrow \mathbb{R}.$$

The closedness of  $\alpha$  implies that this function is constant over all  $x \in \tau \cap \Omega$ . Fix a point  $x \in \sigma \cap \Omega$ , then define  $c_i := f_i(x)$  and  $\alpha' := \sum_{i=1}^k c_i d' v_i$ .

We want to show that  $\alpha$  and  $\alpha'$  are equivalent when restricted to  $\Omega$ . Then we are done because  $\alpha'$  is certainly in the image of  $V^*$ . For any  $\tau \in \mathcal{C}$  and any  $w_\tau \in \Lambda^p \mathbb{L}(\tau)$  write  $w_\tau = \sum_{i=1}^k \lambda_i v_i$ . Then for any  $y \in \Omega$  such that  $y \in \text{int}(\tau)$  we have

$$\begin{aligned} \langle \alpha, w_\tau \rangle(y) &= \langle \alpha, w_\tau \rangle(x) = \sum_{i=1}^k \lambda_i \langle \alpha, v_i \rangle(x) = \sum_{i=1}^k \lambda_i f_i(x) \\ &= \sum_{i=1}^k \lambda_i c_i = \sum_{i=1}^k \lambda_i \langle \alpha', v_i \rangle(y) = \langle \alpha', w_\tau \rangle(y). \end{aligned}$$

The first equality follows because the function defined in (1) is constant. The third equality follows from the fact that  $f_i = \langle \alpha, v_i \rangle$ . Therefore,  $\alpha$  and  $\alpha'$  are equivalent when restricted to  $\Omega$ .

For the general case  $I \neq \emptyset$ , first we apply the above argument to  $\Omega_I$  which is a basic open subset of the polyhedral complex  $\mathcal{C}_I$  by Lemma 3.8. Writing  $X = |\mathcal{C}|$  and  $X_I = |\mathcal{C}_I|$  we obtain

$$\mathcal{L}_{X_I}^p(\Omega_I) = \left( \sum_{\tau \in \mathcal{C}_I : \sigma \prec \tau} \Lambda^p \mathbb{L}(\tau) \right)^*.$$

Thus we only have to show

$$\mathcal{L}_{X_I}^p(\Omega_I) \cong \mathcal{L}_X^p(\Omega).$$

Using the pullbacks of the projection maps define

$$\begin{aligned} \mathcal{L}_{X_I}^p(\Omega_I) &\rightarrow \mathcal{L}_X^p(\Omega) \\ \alpha_I &\mapsto (\pi_{IJ}^* \alpha_I)_{J \subset I}. \end{aligned}$$

This is clearly well defined and injective, we thus have to show surjectivity. More precisely, for  $\alpha \in \mathcal{L}_X^p(\Omega)$ , it remains to show that  $\alpha_J|_{\Omega_J \cap \tau} = \pi_{IJ}^* (\alpha_I|_{\Omega_I \cap \tau})$  for all  $J \subset I$  and  $\tau$  such that  $\sigma \prec \tau$ . By the condition of compatibility for  $\alpha$  there exists a neighborhood  $\Omega_x$  of  $x$  such that

$$\alpha_J|_{\Omega_{x,J}} = \pi_{IJ}^* (\alpha_I|_{\Omega_{x,I}}),$$

hence in particular

$$\alpha_J|_{\Omega_{x,J} \cap \tau} = \pi_{IJ}^* (\alpha_I|_{\Omega_{x,I} \cap \tau}).$$

Since  $\Omega_I \cap \tau$  is connected, the restriction  $\mathcal{L}_{X_I}^p(\Omega_I \cap \tau) \rightarrow \mathcal{L}_{X_I}^p(\Omega_{x,I} \cap \tau)$  is injective and the same for  $J$ , thus we have

$$\alpha_J|_{\Omega_J \cap \tau} = (\pi_{IJ}^* \alpha_I|_{\Omega_I \cap \tau}).$$

The claim concerning the restriction maps is clear if the minimal polyhedra of  $\Omega$  and  $\Omega'$  are of the same sedentarity. If the minimal polyhedron  $\sigma'$  of  $\Omega'$  is of sedentarity  $J$ , then the restriction  $\mathcal{L}_X^p(\Omega) \rightarrow \mathcal{L}_X^p(\Omega')$  is given by restriction on each stratum. It is an exercise to show that identifying  $\mathcal{L}_X^p(\Omega) \cong \mathcal{L}_{X_I}^p(\Omega_I)$  and  $\mathcal{L}_X^p(\Omega') \cong \mathcal{L}_{X_J}^p(\Omega'_J)$  gives the claimed restriction maps.  $\square$

**3.3. Equivalence of cohomologies.** We are now ready to prove that tropical cohomology and Dolbeault cohomology of superforms are isomorphic. We will use the results established in the previous two subsections.

**Lemma 3.17.** *Let  $X$  be a polyhedral space. Then there is a canonical isomorphism of sheaves  $\mathcal{L}_X^p \cong \mathcal{F}_X^p$ .*

*Proof.* Let  $(\varphi_i : U_i \rightarrow \Omega_i \subset X_i)$  be an atlas for  $X$  and choose a polyhedral structure  $\mathcal{C}_i$  on  $X_i$  for all  $i$ . For  $\Omega \subset X$  a basic open subset there is an isomorphism  $\mathcal{L}^p(\Omega) \rightarrow \mathcal{F}^p(\Omega)$  by Proposition 3.16 and Lemma 3.9. Also for  $\Omega' \subset \Omega$  the restriction maps form the following commutative diagram:

$$\begin{array}{ccc} \mathcal{L}^p(\Omega) & \longrightarrow & \mathcal{L}^p(\Omega') \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{F}^p(\Omega) & \longrightarrow & \mathcal{F}^p(\Omega') \end{array}$$

Thus  $\mathcal{L}^p$  and  $\mathcal{F}^p$  agree on a basis of the topology of  $X$ , by [EH00, Proposition I-12, i)] the sheaves agree.  $\square$

We next arrive at Theorem 1 from the introduction.

**Theorem 3.18.** *Let  $X$  be a polyhedral space which is regular at infinity and has a face structure, then there are canonical isomorphisms, induced by the isomorphism  $\mathcal{L}^p \cong \mathcal{F}^p$  above,*

$$\begin{aligned} H_{\text{trop}}^{p,q}(X) &\cong H_{d''}^{p,q}(X) \text{ and} \\ H_{\text{trop},c}^{p,q}(X) &\cong H_{d'',c}^{p,q}(X). \end{aligned}$$

*Proof.* Corollary 3.14 relates the Dolbeault cohomology of superforms with the cohomology of the sheaf  $\mathcal{L}^p$ . Proposition 3.11 does the same with tropical cohomology and the cohomology of  $\mathcal{F}^p$ . Combining this with the Lemma 3.17 proves the isomorphisms.  $\square$

**Remark 3.19.** Notice that in the absence of a face structure on  $X$ , the above argument still relates the sheaf cohomology of  $\mathcal{F}^p$  with the Dolbeault cohomology of superforms. Without a face structure one could also use the sheaves  $\mathcal{F}^p$  to define tropical cohomology for general polyhedral spaces. Then Lemma 3.17 and Corollary 3.14 show that tropical cohomology and Dolbeault cohomology of superforms are still isomorphic in this more general setting.

**Proposition 3.20.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be polyhedral complexes in  $\mathbb{T}^r$  and  $\mathbb{T}^s$  respectively and let  $\delta : |\mathcal{D}| \rightarrow \mathbb{T}^r$  be an extended affine map such that the image of every face of  $\mathcal{C}$  is a face of  $\mathcal{D}$ . Let  $X \subset |\mathcal{C}|$  and  $Y \subset |\mathcal{D}|$  be open subsets such that  $\delta(Y) \subset X$ . Then we have maps  $\delta_{d''}^* : H_{d''}^{p,q}(X) \rightarrow H_{d''}^{p,q}(Y)$  and  $\delta_{\text{trop}}^* : H_{\text{trop}}^{p,q}(X) \rightarrow H_{\text{trop}}^{p,q}(Y)$  and the following diagram commutes:*

$$\begin{array}{ccc} H_{d''}^{p,q}(X) & \longleftrightarrow & H_{\text{trop}}^{p,q}(X) \\ \delta_{d''}^* \downarrow & & \downarrow \delta_{\text{trop}}^* \\ H_{d''}^{p,q}(Y) & \longleftrightarrow & H_{\text{trop}}^{p,q}(Y) \end{array}$$



If  $\delta$  is a proper map, then the same holds for cohomology with compact support.

*Proof.* We have the following commutative diagram:

$$\begin{array}{ccccccc}
 \delta^{-1} \mathcal{A}_X^{p,\bullet} & \longleftarrow & \delta^{-1} \mathcal{L}_X^p & \longleftrightarrow & \delta^{-1} \mathcal{F}_X^p & \longrightarrow & \delta^{-1} \mathcal{C}_X^{p,\bullet} \\
 \delta_{d''}^* \downarrow & & \delta^* \downarrow & & \delta^* \downarrow & & \delta_{\text{trop}}^* \downarrow \\
 \mathcal{A}_Y^{p,\bullet} & \longleftarrow & \mathcal{L}_Y^p & \longleftrightarrow & \mathcal{F}_Y^p & \longrightarrow & \mathcal{C}_Y^{p,\bullet}
 \end{array}$$

All horizontal maps are quasi-isomorphisms. Now taking hypercohomology of the functor of global sections, we get the cohomology of the complexes of global sections on the far left and right and the cohomology of the sheaves in the middle. This shows the commutativity of the left square in

$$\begin{array}{ccccc}
 H_{d''}^{p,q}(X) & \longleftrightarrow & H^q(\mathcal{C}_X^{p,\bullet}(X), \partial^*) & \longleftrightarrow & H_{\text{trop}}^{p,q}(X) \\
 \delta_{d''}^* \downarrow & & \delta_{\text{trop}}^* \downarrow & & \delta_{\text{trop}}^* \downarrow \\
 H_{d''}^{p,q}(Y) & \longleftrightarrow & H^q(\mathcal{C}_Y^{p,\bullet}(Y), \partial^*) & \longleftrightarrow & H_{\text{trop}}^{p,q}(Y).
 \end{array}$$

The commutativity of the right square is obvious. If  $\delta$  is proper then the pullbacks are well defined for sections with compact support and the same arguments work.  $\square$

**Remark 3.21.** For a polyhedral space  $X$  it is easy to see that we have  $\mathcal{L}_X^0 = \mathbb{R} = \mathcal{F}_X^0$ , where  $\mathbb{R}$  is the constant sheaf with stalks  $\mathbb{R}$ . Thus we have  $H_{d''}^{0,q}(X) \cong H_{\text{sing}}^q(X) \cong H_{\text{trop}}^{0,q}(X)$  by Proposition 3.11, Corollary 3.14 and [Bre97, Chapter III, Theorem 1.1].

The tropical cohomology groups and Dolbeault cohomology groups of superforms for  $p > 0$  do however depend heavily on the equivalence class of the chosen atlas, and not just on the topological space underlying a polyhedral space (cf. Example 3.15).

**Remark 3.22.** Another technique to prove Theorem 3.18 could be to use a map similar to the de Rham map, which provides an isomorphism between the de Rham and singular cohomologies in the classical theory. This de Rham map is given explicitly by

$$\begin{aligned}
 H_{dR}^q(X) &\rightarrow H_{\text{sing}}^q(X), \\
 [\alpha] &\mapsto \left( (\delta : \Delta_q \rightarrow X) \mapsto \int_{\Delta} \delta^*(\alpha) \right).
 \end{aligned}$$

We could do something similar for Dolbeault cohomology of superforms and tropical cohomology, since contracting a  $(p, q)$ -superform with the coefficient of a cell gives us a  $(0, q)$ -superform which we can then integrate over the simplex. Some care needs to be taken as to allow only smooth simplices (as in the classical case) and also to ensure that the integrals are well defined. We will not do this since it is not required for our considerations. Note that this could also be used to identify the wedge product on Dolbeault cohomology of superforms with the cup product on tropical cohomology.

#### 4. POINCARÉ DUALITY

In this section we prove Poincaré duality for a certain class of polyhedral spaces, namely tropical manifolds. By this we mean an explicit isomorphism  $\text{PD} : H^{p,q}(X) \rightarrow H_c^{n-p, n-q}(X)^*$ . Just as for standard differential forms, this map will be defined using a pairing given by integration of superforms. In Subsection 4.1, we show that the pairing given by integration of superforms descends to a pairing on the cohomology for tropical spaces. Finally we show Poincaré duality for tropical manifolds in Subsection 4.2.

**4.1. Integration of superforms.** In this subsection we consider the standard lattice  $\mathbb{Z}^r \subset \mathbb{R}^r$ . Notice that in  $\mathbb{T}^r$  there is an induced lattice in each stratum  $\mathbb{R}_I^r$ . Integration of superforms on rational polyhedral complexes in  $\mathbb{T}^r$  (cf. Definition 4.2) is an extension of the theory already developed in  $\mathbb{R}^r$ . Using partitions of unity, superforms on rational polyhedral spaces can also be integrated. Theorem 4.7 is a version of Stokes' theorem for superforms on tropical spaces (cf. Definition 4.6), which ensures that the pairing defined by the wedge product and integration descends to cohomology.

**Lemma 4.1.** *Let  $X$  be a polyhedral subspace of dimension  $n$  in  $\mathbb{T}^r$  that is the closure of a polyhedral subspace  $X_I$  of dimension  $n$  in  $\mathbb{R}_I^r$ . If  $\alpha \in \mathcal{A}_c^{p,q}(X)$  is such that  $\max(p, q) = n$ , then  $\alpha_I \in \mathcal{A}^{p,q}(X_I)$  has compact support and for each  $J \supsetneq I$  we have  $\alpha_J = 0$ .*

*Proof.* We may assume that  $I = \emptyset$ . Recall that  $\text{supp}(\alpha) = \cup_{J \subset [r]} \text{supp}(\alpha_J)$ . We will show that  $\text{supp}(\alpha_\emptyset)$  is a closed subset of the compact set  $\text{supp}(\alpha)$ , thus also compact. Since  $X \subset \mathbb{T}^r$  is the closure of  $X_I$  for any  $J \supsetneq I$  we have  $\dim(X_J) < n$ , and the  $(p, q)$ -superform  $\alpha_J$  with  $\max(p, q) = n$  must be zero.

For  $x \in \mathbb{T}^r$  of sedentarity  $J \neq \emptyset$  let  $\Omega_x$  be such that  $\alpha_\emptyset|_{\Omega_x} = \pi_J^*(\alpha_J|_{\Omega_{x,J}}) = 0$ . Then

$$\text{supp}(\alpha_\emptyset) \subset \text{supp}(\alpha) \setminus \bigcup_{x: \text{sed}(x) \neq \emptyset} \Omega_x$$

hence  $\text{supp}(\alpha_\emptyset)$  is a closed subset of a compact set, thus also compact.  $\square$

**Definition 4.2.** A polyhedral complex  $\mathcal{C}$  in  $\mathbb{T}^r$  is called *rational* if every polyhedron  $\sigma$  is parallel to a subspace of  $\mathbb{R}_{\text{sed}(\sigma)}^r$  defined over  $\mathbb{Z}$ . Then for any polyhedron  $\sigma$  there is a canonical lattice of full rank  $\mathbb{Z}(\sigma) \subset \mathbb{L}(\sigma)$ . A *weighted polyhedral subspace* is a pure dimensional polyhedral complex equipped with integer valued weights on its top dimensional facets up to common refinements preserving the weights. Call a weighted polyhedral complex  $\mathcal{C}$  representing  $X$  a *weighted polyhedral structure* on  $X$ . A *weighted rational polyhedral space* is a polyhedral space such that the targets of all of its charts are weighted rational polyhedral subspaces and the transition maps are integer affine and weight preserving.

We recall the definition of integration of superforms on polyhedral complexes in  $\mathbb{R}^r$  from [CLD12] (see also [Gub13], whose notation we follow) and extend the definitions to polyhedral complexes in  $\mathbb{T}^r$ .

**Definition 4.3.** Let  $\mathcal{C}$  be a weighted rational polyhedral complex in  $\mathbb{R}^r$ , which is pure of dimension  $n$ . We write  $\mathcal{C}_n$  for the set of  $n$ -dimensional polyhedra.

- i) Let  $\alpha \in \mathcal{A}_c^{n,n}(|\mathcal{C}|)$ . For  $\sigma \in \mathcal{C}_n$ , choose a basis  $x_1, \dots, x_n$  of  $\mathbb{Z}(\sigma)$ . Then  $\alpha|_\sigma$  can be written as

$$f_\alpha d'x_1 \wedge d''x_1 \wedge \dots \wedge d''x_n = (-1)^{\frac{n(n-1)}{2}} f_\alpha d'x_1 \wedge d'x_2 \wedge \dots \wedge d''x_{n-1} \wedge d''x_n$$

for  $f_\alpha \in \mathcal{A}_c^{0,0}(\sigma)$ . Note that, since this is an integral basis,  $f_\alpha$  is independent of the choice of  $x_1, \dots, x_n$ . Then the *integral of  $\alpha$  over  $\sigma$*  is

$$\int_\sigma \alpha := \int_\sigma f_\alpha,$$

where the integral on the right is taken with respect to the volume defined by the lattice  $\mathbb{Z}(\sigma) \subset \mathbb{L}(\sigma)$ . The integral over the weighted rational polyhedral complex  $\mathcal{C}$  is

$$\int_{\mathcal{C}} \alpha := \sum_{\sigma \in \mathcal{C}_n} m_\sigma \int_\sigma \alpha,$$

where  $m_\sigma$  is the weight of  $\sigma$ .

- ii) Let  $\tau \prec \sigma$  be a face of  $\sigma$  of codimension 1. Denote by  $\nu_{\tau,\sigma} \in \mathbb{Z}(\sigma)$  a representative of the unique generator of  $\mathbb{Z}(\sigma)/\mathbb{Z}(\tau)$  which points inside of  $\sigma$ . Then for  $\beta \in \mathcal{A}_c^{n,n-1}(|\mathcal{C}|)$  the *boundary integral of  $\beta$  over  $\partial\sigma$*  is

$$\int_{\partial\sigma} \beta = \sum_{\tau \prec \sigma} \int_{\tau} \langle \beta; \nu_{\tau,\sigma} \rangle_n,$$

where on the right hand side we use the integral of the  $(n-1, n-1)$ -form  $\langle \beta; \nu_{\tau,\sigma} \rangle_n$  over the  $(n-1)$ -dimensional polyhedron  $\tau$  as defined in *i*). The integral over the boundary of the weighted rational polyhedral subspace  $\mathcal{C}$  is

$$\int_{\partial\mathcal{C}} \beta := \sum_{\sigma \in \mathcal{C}_n} m_{\sigma} \int_{\partial\sigma} \beta,$$

where  $m_{\sigma}$  is the weight of  $\sigma$ .

- iii) If  $\mathcal{C}$  is a weighted rational polyhedral complex in  $\mathbb{T}^r$ , then the definitions from *i*) and *ii*) can be extended. Note that if  $\alpha \in \mathcal{A}_c^{p,q}(|\mathcal{C}|)$  and  $\sigma \in \mathcal{C}_n$  is the closure of  $\sigma' \in \mathcal{C}_{I,n}$ , then the support of  $\alpha|_{\sigma}$  is contained in  $\sigma'$  by Lemma 4.1 and we define

$$\int_{\sigma} \alpha := \int_{\sigma'} \alpha|_{\sigma'}.$$

The same works for integrals over codimension 1 faces. Note that if the codimension 1 face  $\tau$  of  $\sigma$  is of a higher sedentarity than  $\sigma$ , then, again by Lemma 4.1, for  $\beta \in \mathcal{A}_c^{n,n-1}(|\mathcal{C}|)$  the restriction  $\beta|_{\sigma}$  has support away from  $\tau$ . It thus suffices for  $\int_{\partial\sigma} \beta$  to consider only codimension 1 faces of the same sedentarity as  $\sigma$ .

- iv) For a weighted rational polyhedral subspace  $X$  we define

$$\int_X \alpha := \int_{\mathcal{C}} \alpha,$$

where  $\mathcal{C}$  is a weighted rational polyhedral structure on  $X$ . This is well defined since the integral is invariant under subdivision.

- v) Let  $\alpha \in \mathcal{A}_c^{n,n}(\Omega)$  for  $\Omega$  an open subset of an  $n$ -dimensional weighted rational polyhedral subspace  $X$  in  $\mathbb{T}^r$ . Then  $\alpha$  can be extended by zero to a form  $\alpha \in \mathcal{A}_c^{n,n}(X)$ . Hence we can integrate superforms with compact support on open subsets of a polyhedral subspace over the entire polyhedral subspace.

Now integration on polyhedral spaces can be defined by using the integration on polyhedral subspaces and partitions of unity just as in manifold theory.

**Definition 4.4.** Let  $X$  be an  $n$ -dimensional weighted rational polyhedral space with atlas  $(\varphi_i : U_i \rightarrow \Omega_i \subset X_i)_{i \in I}$ . Let  $\alpha \in \mathcal{A}_c^{n,n}(X)$  and  $(f_j)_{j \in J}$  be a partition of unity with functions in  $\mathcal{A}^{0,0}$  subordinate to the cover  $(U_i)$  as in Lemma 2.7. Then we have

$$\alpha = \sum_{j \in J} f_j \alpha,$$

which is a finite sum. Define  $\alpha_j \in \mathcal{A}_c^{n,n}(\Omega_i)$  the superform corresponding to  $f_j \alpha \in \mathcal{A}_c^{n,n}(U_i)$ . As mentioned the superform  $\alpha_j$  can be extended to a superform in  $\mathcal{A}^{n,n}(X_i)$ . Then the integral of  $\alpha$  over  $X$  is

$$\int_X \alpha := \sum_{j \in J} \int_{X_i} \alpha_j,$$

with the integral on the right as defined in Definition 4.3.

The following lemma ensures the above defined integral is independent of the choice of charts and partition of unity on the polyhedral space.

**Lemma 4.5.** *Let  $X \subset \mathbb{T}^r$  and  $X' \subset \mathbb{T}^{r'}$  be weighted rational polyhedral subspaces, let  $\Omega \subset X$  and  $\Omega' \subset X'$  be open subsets.*

*Let  $F : \Omega' \rightarrow \Omega$  and  $G : \Omega \rightarrow \Omega'$  be extended integral affine maps such that  $F \circ G = \text{id}_\Omega$ ,  $G \circ F = \text{id}_{\Omega'}$  and  $F$  and  $G$  preserve weights. Then for  $\alpha \in \mathcal{A}_c^{n,n}(\Omega)$  we have  $\int_\Omega \alpha = \int_{\Omega'} F^* \alpha$ .*

*Proof.* It is sufficient to consider the situation for polyhedral subspaces of  $\mathbb{R}^r$ , since the pullback of a superform  $\alpha$  on a polyhedral complex in  $\mathbb{T}^r$  is defined by pulling back the components  $\alpha_I$  along affine maps and integration is also defined by considering the intersection of the polyhedral subspace with the vector spaces  $\mathbb{R}_I^r$ . After possibly shrinking  $X$  outside of  $\Omega'$  we may assume that there exists a polyhedral structure  $\mathcal{C}'$  on  $X'$  such that all maximal polyhedra intersect  $\Omega'$ . Let  $F_* \mathcal{C}'$  be the push-forward of  $\mathcal{C}'$  in the sense of weighted polyhedral complexes (cf. [Gub13, 3.9]). Note that then  $F$  extends to a map  $F : |\mathcal{C}'| \rightarrow |F_* \mathcal{C}'|$  whose inverse is given by an extension of  $G$ . We have  $\int_{\Omega'} F^* \alpha = \int_{\mathcal{C}'} F^* \alpha$  and, since  $F$  preserves weights, also  $\int_\Omega \alpha = \int_{F_* \mathcal{C}'} \alpha$ . Now the result follows from the projection formula [Gub13, Proposition 3.10].  $\square$

**Definition 4.6.** Let  $\mathcal{C}$  be a weighted rational polyhedral complex in  $\mathbb{R}^r$  which is pure of dimension  $n$ . Let  $\tau$  be a face of  $\mathcal{C}$  of dimension  $n - 1$ . We say that  $\mathcal{C}$  is *balanced at  $\tau$*  if

$$\sum_{\sigma \in \mathcal{C}_n : \tau \prec \sigma} m_\sigma \nu_{\sigma, \tau} \in \mathbb{Z}(\tau),$$

with  $\nu_{\sigma, \tau}$  as introduced in Definition 4.3 ii). We say that  $\mathcal{C}$  satisfies the *balancing condition* if it is balanced at every face of dimension  $n - 1$ .

A *tropical cycle*  $X$  of dimension  $n$  in  $\mathbb{R}^r$  is a weighted rational polyhedral subspace of pure dimension  $n$ , which admits a weighted rational polyhedral structure  $\mathcal{C}$  satisfying the balancing condition. If the integer weights of  $X$  are all positive, then  $X$  is a *tropical variety*.

A tropical cycle  $X \subset \mathbb{T}^n$  of sedentarity  $I$  is the closure in  $\mathbb{T}^n$  of a tropical cycle  $X^\circ \subset \mathbb{R}_I^n \cong \mathbb{R}^{n-|I|}$ .

A *tropical space*  $X$  is a polyhedral space such that in each chart  $\varphi_i : U_i \rightarrow \Omega_i \subset X_i$ , the space  $X_i$  is a tropical cycle of sedentarity  $\emptyset$  with the condition that the weight functions on the faces of  $X_i$  and  $X_j$  are consistent on the overlaps  $U_i$  and  $U_j$ . If the integer weights of  $\Omega_i$  are all positive, then we call  $X$  an *effective tropical space*.

**Theorem 4.7** (Stokes' theorem for tropical spaces). *Let  $X$  be a weighted  $n$ -dimensional rational polyhedral space. Then  $X$  is a tropical space if and only if for all  $\beta \in \mathcal{A}_c^{n,n-1}(X)$  we have*

$$\int_X d'' \beta = 0.$$

*Proof.* The case of a weighted rational polyhedral subspace  $X \subset \mathbb{T}^r$  is a direct consequence of Definition 4.3 iii) and [Gub13, Proposition 3.8]. The statement weighted rational polyhedral and tropical spaces follows immediately.  $\square$

**Definition 4.8.** Let  $X$  be a tropical space of dimension  $n$ . There is a product

$$\begin{aligned} \mathcal{A}^{p,q}(X) \times \mathcal{A}_c^{n-p,n-q}(X) &\rightarrow \mathbb{R}, \\ (\alpha, \beta) &\mapsto \int_X \alpha \wedge \beta. \end{aligned}$$

By Stokes' theorem 4.7 we have for  $\alpha \in \mathcal{A}^{p,q}(X)$  and  $\beta \in \mathcal{A}_c^{n-p,n-q-1}(X)$  the following

$$0 = \int_X d''(\alpha \wedge \beta) = \int_X d'' \alpha \wedge \beta + \int_X (-1)^{p+q} \alpha \wedge d'' \beta.$$

Thus

$$\int_X d'' \alpha \wedge \beta = (-1)^{p+q+1} \int_X \alpha \wedge d'' \beta.$$

We define

$$\begin{aligned} \text{PD} : \mathcal{A}^{p,q}(X) &\rightarrow \mathcal{A}_c^{n-p,n-q}(X)^*, \\ \alpha &\mapsto \left( \beta \mapsto \varepsilon \int_X \alpha \wedge \beta \right) \end{aligned}$$

where  $\mathcal{A}_c^{n-p,n-q}(X)^* := \text{Hom}_{\mathbb{R}}(\mathcal{A}_c^{n-p,n-q}(X), \mathbb{R})$  denotes the (non-topological) dual vector space of  $\mathcal{A}_c^{n-p,n-q}(X)$  and  $\varepsilon = (-1)^{p+q/2}$  if  $q$  is even and  $\varepsilon = (-1)^{(q+1)/2}$  if  $q$  is odd. Our choice of  $\varepsilon$  together with the Leibniz rule and Stokes' theorem implies that we have a morphism of complexes

$$\text{PD} : \mathcal{A}^{p,\bullet}(X) \rightarrow \mathcal{A}_c^{n-p,n-\bullet}(X)^*,$$

where the dual complex is equipped with the dual differential. We now get a map in cohomology

$$\text{PD} : H_{d''}^{p,q}(X) \rightarrow H_{d'',c}^{n-p,n-q}(X)^*,$$

since we have

$$H^q(\mathcal{A}_c^{n-p,n-\bullet}(X)^*, d''^*) = (H_q(\mathcal{A}_c^{n-p,n-\bullet}(X), d''))^* = H_{d'',c}^{n-p,n-q}(X)^*.$$

We will show that PD is an isomorphism for tropical manifolds in the next section.

**4.2. Poincaré duality for tropical manifolds.** In this section we consider only polyhedral spaces which are regular at infinity, and we write  $H^{p,q}(X)$  for  $H^q(X, \mathcal{L}^p)$ . By Corollary 3.14, we also have  $H^{p,q}(X) = H_{d''}^{p,q}(X)$ . If  $X$  has a face structure, it can be used to define the tropical cohomology of  $X$  and by Theorem 3.18 we have canonical isomorphisms

$$H^{p,q}(X) \cong H^q(X, \mathcal{L}^p) \cong H^q(X, \mathcal{F}^p) \cong H_{\text{trop}}^{p,q}(X).$$

Similarly, in the setting of cohomology with compact support, let  $H_c^{p,q}(X) := H_c^q(X, \mathcal{L}^p)$ . We will show that the Poincaré duality map given in Definition 4.8 is an isomorphism for tropical manifolds.

**Definition 4.9.** Let  $X$  be an  $n$ -dimensional tropical space. We say that  $X$  has *Poincaré duality* (PD) if the Poincaré duality map

$$\text{PD} : H^{p,q}(X) \rightarrow H_c^{n-p,n-q}(X)^*$$

is an isomorphism for all  $p, q$ .

**Example 4.10.** Consider again the polyhedral spaces from Example 3.15. In these examples, the underlying topological space is  $[0, 1]$ , hence compact. Therefore, cohomology and cohomology with compact support are isomorphic. If we take the charts for  $X$  in such a way that  $[0, 1]$  is the glueing of two copies of  $\mathbb{T}^1$  (cf. Example 2.22), then we have by Example 3.15 the equalities  $h^{0,0}(X) = 1 = h^{1,1}(X)$  and  $h^{1,0}(X) = 0 = h^{0,1}(X)$ . If we put weight 1 on  $\mathbb{T}^1$  in both charts, then  $X$  becomes an effective tropical space. It is easy to see that the integration pairing  $H^{0,0}(X) \times H^{1,1}(X) \rightarrow \mathbb{R}$  is non-degenerate and so  $X$  has PD.

The polyhedral space defined by taking a single chart on  $[0, 1]$  given by the inclusion of the interval into  $\mathbb{R}$ , with  $[0, 1]$  equipped with weight 1, does not satisfy Stokes' theorem. Thus the PD map is not defined on cohomology. We already saw in Example 3.15 that the dimensions of the respective cohomology groups do not agree. There are also examples of tropical spaces which do not satisfy PD. Take for example  $Y$  to be the union of the coordinate axes in  $\mathbb{R}^2$ , again with weight 1 on each facet. Then it is clear that  $h^{0,0}(Y) = 1$ , since  $H^{0,0}(Y)$  is the usual cohomology group  $H^0(Y; \mathbb{R})$ . However, it can be shown that  $h_c^{1,1}(Y) = 2$ .

The rest of this section is devoted to proving Theorem 2, which states that tropical manifolds have Poincaré duality. Tropical manifolds are tropical spaces locally modeled on matroidal fans, see Definition 4.11 below. Matroids are a combinatorial abstraction of the notion of independence in mathematics. See [Oxl11] for a comprehensive introduction to the theory of matroids. Every matroid has a representation as a fan tropical cycle. [Stu02]. Ways of explicitly constructing this fan can be found in [FS05], [AK06]. Here matroidal fans are always equipped with weights 1 on all facets.

**Definition 4.11.** A *tropical manifold* is a tropical space  $X$  of dimension  $n$  such that there is an atlas  $A = (\varphi_i : U_i \rightarrow \Omega_i \subset X_i)$ , such that all  $X_i$  are of the form  $\mathbb{T}^{r_i} \times V_i$  for matroidal fans  $V_i$  of dimension  $n - r_i$  in  $\mathbb{R}^{s_i}$ .

Notice that tropical manifolds are polyhedral spaces which are regular at infinity.

We begin by showing that matroidal fans in  $\mathbb{R}^r$  have Poincaré duality. To do this we use an alternative recursive description of matroidal fans via an operation known as tropical modifications (see [BIMS15] for an introduction). In the language of matroids, the operation of tropical modification is related to deletions and contractions.

We now describe the operation of tropical modification. Let  $W \subset \mathbb{R}^{r-1}$  be a tropical variety and  $P : \mathbb{R}^{r-1} \rightarrow \mathbb{R}$  a piecewise integer affine function. The graph  $\Gamma_P(W) \subset \mathbb{R}^r$  is a rational polyhedral complex, which inherits weights from the weights of  $W$ . In general, this graph is not a tropical cycle, since it does not satisfy the balancing condition, because  $P$  is only piecewise linear. However, the graph  $\Gamma_P(W)$  can be completed to a tropical cycle  $V$  in a canonical way; at a codimension 1 face  $E$  of  $\Gamma_P(W)$  not satisfying the balancing condition attach a facet to  $E$  generated by the direction  $-e_r$ . Then this facet can be equipped with a unique integer weight so that the resulting polyhedral complex is now balanced at  $E$ . Applying this procedure at all codimension one facets of  $\Gamma_P(W)$  produces a tropical cycle  $V$ . Notice that there is a map  $\delta : V \rightarrow W$  induced by the linear projection. The divisor of the piecewise integer affine function  $P$  restricted to  $W$  is a tropical cycle  $\text{div}_W(P) \subset W$  which is supported on the points  $w \in W$  such that  $\delta^{-1}(w)$  is a half-line. The weights on the facets of  $\text{div}_W(P)$  are inherited from the weights of  $V$ .

**Definition 4.12.** Let  $W \subset \mathbb{R}^{r-1}$  be a tropical cycle and  $P : \mathbb{R}^r \rightarrow \mathbb{R}$  a piecewise integer affine function, then the *open tropical modification* of  $W$  along  $P$  is the map  $\delta : V \rightarrow W$  where  $V \subset \mathbb{R}^r$  is the tropical cycle described above. A *closed tropical modification* is a map  $\bar{\delta} : \bar{V} \rightarrow W$  where  $\bar{V} \subset \mathbb{R}^{r-1} \times \mathbb{T}$  is the closure of  $V$  and  $\bar{\delta}$  is the extension of  $\delta$ .

A *matroidal tropical modification* is a modification where  $V, W$ , and  $\text{div}_W(P)$  are all matroidal fans.

Note that for a closed tropical modification  $\bar{\delta} : \bar{V} \rightarrow W$  with divisor  $D$ , we have that  $\bar{\delta}|_{\bar{V}_r} : \bar{V}_r \rightarrow D$  identifies the subspace  $\bar{V}_r = \{x \in V \mid x_r = -\infty\}$  with  $D$ . We thus may also consider  $D$  as a subspace of  $\bar{V}$ .

It follows from the next proposition that for any matroidal fan  $V \subset \mathbb{R}^r$  of dimension  $n$  there is a sequence of open matroidal tropical modifications  $V \rightarrow W_1 \rightarrow \cdots \rightarrow W_{r-n} = \mathbb{R}^n$ .

**Proposition 4.13.** [Sha13b, Proposition 2.25] *Let  $V \subsetneq \mathbb{R}^r$  be a matroidal fan, then there is a coordinate direction  $e_i$  such that the linear projection  $\delta : \mathbb{R}^r \rightarrow \mathbb{R}^{r-1}$  with kernel generated by  $e_i$  is a matroidal tropical modification  $\delta : V \rightarrow W$  along a piecewise integer affine function  $P$ , i.e.  $W \subset \mathbb{R}^{r-1}$  and  $D = \text{div}_W(P) \subset \mathbb{R}^{r-1}$  are matroidal fans.*

Tropical cohomology is invariant under closed tropical modifications [Sha15, Theorem 4.13]. The next lemma checks that these isomorphisms also apply to cohomology with compact support and that they are compatible with the PD map.

**Proposition 4.14.** *Let  $\delta : \bar{V} \rightarrow W$  be a closed tropical modification of matroidal fans where  $W \subset \mathbb{R}^{r-1}$  and  $\bar{V} \subset \mathbb{R}^{r-1} \times \mathbb{T}$ . Then there are isomorphisms*

$$\delta^* : H^{p,q}(W) \rightarrow H^{p,q}(\bar{V}) \quad \text{and} \quad \delta^* : H_c^{p,q}(W) \rightarrow H_c^{p,q}(\bar{V}),$$

which are induced by the pullback of superforms and are compatible with the Poincaré duality map.

*Proof.* The fact that  $\delta^*$  is an isomorphism for tropical cohomology is shown in [Sha15, Theorem 4.13]. By Proposition 3.20 this also applies to  $H^{p,q}$ . The same arguments as used in [Sha15, Theorem 4.13] work for cohomology with compact support, since  $\delta$  and the homotopy used there are proper maps. Thus again by Proposition 3.20, this also applies to  $H_c^{p,q}$ .

To show that the isomorphism  $\delta^*$  is compatible with the Poincaré duality map, it suffices to show that for  $\omega \in \mathcal{A}_c^{n,n}(W)$  we have

$$\int_W \omega = \int_{\overline{V}} \delta^*(\omega),$$

since the wedge product is compatible with the pullback. The fan  $W$  is the push-forward of  $\overline{V} \cap \mathbb{R}^r$  along  $\delta$  in the sense of polyhedral complexes and then the result follows from the projection formula [Gub13, Proposition 3.10] because the support of  $\delta^{-1}(\omega)$  is contained in  $\overline{V} \cap \mathbb{R}^r$  by Lemma 4.1.  $\square$

The next statements relate the cohomology groups of the matroidal fans appearing in an open tropical modification by exact sequences.

**Proposition 4.15.** *Let  $\Omega$  be an open subset of a polyhedral subspace in  $\mathbb{T}^r$ . Let  $i \in [r]$  and write  $D = \Omega \cap \mathbb{T}_i^r$  and  $U := \Omega \setminus D$ . Then there exists a long exact sequence in cohomology with compact support*

$$\dots \rightarrow H_c^{p,q-1}(D) \rightarrow H_c^{p,q}(U) \rightarrow H_c^{p,q}(\Omega) \rightarrow H_c^{p,q}(D) \rightarrow H_c^{p,q+1}(U) \rightarrow \dots$$

*Proof.* Note first that  $U$  and  $\Omega$  are polyhedral spaces via the inclusion into  $\mathbb{T}^r$ . Further  $D$  is a polyhedral space via the inclusion into  $\mathbb{T}_i^r$ . We claim that the natural sequence of complexes

$$0 \rightarrow \mathcal{A}_{\Omega,c}^{p,\bullet}(U) \rightarrow \mathcal{A}_{\Omega,c}^{p,\bullet}(\Omega) \rightarrow \mathcal{A}_{D,c}^{p,\bullet}(D) \rightarrow 0$$

is exact. By the condition of compatibility for superforms along strata, if a superform restricts to 0 on  $D$ , then it must be 0 on a neighborhood of  $D$ . This shows exactness in the middle of the short exact sequence. Both surjectivity of the last map and injectivity of the first map are clear. The result then follows by the long exact cohomology sequence.  $\square$

If  $\delta : V \rightarrow W$  is a tropical modification with divisor  $D$ , upon writing  $U := V$  and  $\Omega := \overline{V}$  we can apply Proposition 4.15, where we identify  $D$  with the subspace of  $\overline{V}$  given by  $\overline{V} \cap \mathbb{T}_r^r$ , as explained above. Together with Proposition 4.14, we can relate the cohomology with compact support of  $V$  with the ones of  $W$  and  $D$ .

**Lemma 4.16.** *Let  $V \subset \mathbb{R}^r$  be a matroidal fan of dimension  $n$ . Then for all  $p$*

$$H_c^{p,q}(V) = 0 \text{ if } q \neq n.$$

*Proof.* The lemma is proven by induction on  $r$ , which is the dimension of the surrounding space. When  $r = 0$  the assertion is clear. We now argue from  $r - 1$  to  $r$ : If  $r = n$ , we are in the case  $V = \mathbb{R}^n$ , since  $\mathbb{R}^n$  is the only matroidal fan of dimension  $n$  in  $\mathbb{R}^n$ . Then we have  $H_c^{p,q}(\mathbb{R}^n) = \Lambda^p \mathbb{R}^{n*} \otimes H_c^q(\mathbb{R}^n)$ , where  $H_c^q(\mathbb{R}^n)$  denotes the usual de Rham cohomology with compact support of  $\mathbb{R}^n$ . We have  $H_c^q(\mathbb{R}^n) = 0$  unless  $q = n$ , thus we have the statement in this case. Otherwise  $r > n$  and we can apply Proposition 4.13. Thus there exists a matroidal fan  $W$  and a tropical modification  $\delta : V \rightarrow W$  whose divisor  $D \subset W$  is a matroidal fan. Now by the induction assumption,  $H_c^{p,q}(D) = 0$  unless  $q = n - 1$  and  $H^{p,q}(W) = 0$  unless  $q = n$ . Applying the long exact sequence from Proposition 4.15 and replacing  $H_c^{p,q}(\overline{V})$  with  $H^{p,q}(W)$  by Proposition 4.14 we have

$$\dots \rightarrow H_c^{p,q-1}(D) \rightarrow H_c^{p,q}(V) \rightarrow H_c^{p,q}(W) \rightarrow \dots,$$

thus we obtain that  $H_c^{p,q}(V) = 0$  if  $q \neq n$  and the lemma is proven.  $\square$

The following short exact sequence involving the  $(p, 0)$ -cohomology groups of matroidal fans is a consequence of a short exact sequence for Orlik-Solomon algebras of matroids [OT92]. For the translation to our setting see [Sha11, Lemma 2.2.7] and [Zha13]. Recall the contraction of superforms we defined in Definition 2.9.

**Lemma 4.17.** *Let  $V \subset \mathbb{R}^{r+1}$  and  $W \subset \mathbb{R}^r$  be matroidal fans and  $\delta : V \rightarrow W$  be an open tropical modification along a divisor  $D \subset W$  which is a matroidal fan, then*

$$0 \longrightarrow H^{p,0}(W) \longrightarrow H^{p,0}(V) \xrightarrow{\langle \cdot ; e_i \rangle_p} H^{p-1,0}(D) \longrightarrow 0$$

*is an exact sequence.*

It follows from [Sha11] that the map  $H^{p,0}(V) \rightarrow H^{p-1,0}(D)$  in the exact sequence above is induced by the contraction  $\langle \cdot ; e_i \rangle_p$  with  $e_i$  in the  $p$ -th component, where the vector  $e_i$  generates the kernel of the linear projection giving the map  $\delta : V \rightarrow W$ . Note that this map is not induced by a map on the level of forms. However for a closed  $(p, 0)$ -form  $\alpha$ , the form  $\langle \alpha ; e_i \rangle_p \in \mathcal{L}^{p-1,0}(V)$  is then the restriction of a unique form in  $\mathcal{L}^{p-1,0}(\overline{V})$  and we can restrict this form to  $D$ , since we can identify  $D$  with a subspace of  $\overline{V}$ . This is easy to see once we use the identifications from Proposition 3.16.

**Lemma 4.18.** *Let  $\delta : V \rightarrow W$  be an open tropical modification of matroidal fans  $V \subset \mathbb{R}^{r+1}$  and  $W \subset \mathbb{R}^r$  along a divisor  $D \subset W$  which is a matroidal fan. Then the diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{p,0}(W) & \longrightarrow & H^{p,0}(V) & \xrightarrow{\langle \cdot ; e_i \rangle_p} & H^{p-1,0}(D) \longrightarrow 0 \\ & & \downarrow \text{PD} & & \downarrow \text{PD} & & \downarrow (-1)^{n-1} \text{PD} \\ 0 & \longrightarrow & H_c^{n-p,n}(W)^* & \longrightarrow & H_c^{n-p,n}(V)^* & \xrightarrow{g^*} & H_c^{n-p,n-1}(D)^* \longrightarrow 0, \end{array}$$

*which is obtained by the exact sequences in Proposition 4.15 and Lemma 4.17, is commutative.*

*Proof.* Note that by Proposition 4.14 the statement is equivalent to the one we obtain when replacing  $W$  by  $\overline{V}$ . Then the fact that the first square commutes is immediate. The map  $g : H_c^{n-p,n-1}(D) \rightarrow H_c^{n-p,n}(V)$  is the boundary operator in a long exact cohomology sequence. We recall its construction: For a closed superform  $\beta \in \mathcal{A}_c^{n-p,n-1}(D)$ , take any lift  $l(\beta) \in \mathcal{A}_c^{n-p,n-1}(\overline{V})$  such that  $l(\beta)|_D = \beta$ . Then  $d''(l(\beta))$  restricts to 0 on  $D$  and thus is a superform with compact support on  $V$ . Then  $g(\beta)$  is given by the class of  $d''(l(\beta))$  in  $H_c^{n-p,n}(V)$ . As usual this does not depend on the choice of  $l(\beta)$ . We have to show that for all closed forms  $\alpha \in \mathcal{A}^{p,0}(V)$  and  $\beta \in \mathcal{A}_c^{n-p,n-1}(V)$  we have

$$(-1)^p \int_V \alpha \wedge d''(l(\beta)) = (-1)^{n+p} \int_D \langle \alpha ; e_i \rangle_p \wedge \beta$$

for some lift  $l(\beta)$ , where  $e_i$  is the coordinate direction of the modification. Let  $P$  be the piecewise linear function of the modification and  $P' = P - 1$ . The graph of  $P'$  divides  $V$  into two polyhedral complexes, one living above the graph and the other one below, which we denote by  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , respectively. Equip all facets of both polyhedral complexes  $\mathcal{C}_1$  and  $\mathcal{C}_2$  with weight 1. Note that  $\delta(|\mathcal{C}_2|) \subset D$ . We find a lift  $l(\beta) \in \mathcal{A}_c^{n-p,n-1}(\overline{V})$  such that  $l(\beta)|_{|\mathcal{C}_2|} = (\delta|_{|\mathcal{C}_2|})^*(\beta)$ . Then we have

$$\int_V \alpha \wedge d''(l(\beta)) = \int_{\mathcal{C}_1} \alpha \wedge d''(l(\beta)) + \int_{\mathcal{C}_2} \alpha \wedge d''(l(\beta)) = \int_{\mathcal{C}_1} \alpha \wedge d''(l(\beta)).$$

By Stokes' theorem in the version [Gub13, Proposition 3.5] and the Leibniz rule we have

$$\int_{\mathcal{C}_1} \alpha \wedge d''(l(\beta)) = (-1)^p \int_{\partial \mathcal{C}_1} \alpha \wedge l(\beta).$$

It follows from the proof of [Gub13, Theorem 3.8] that the boundary integral of  $\alpha \wedge l(\beta)$  over balanced codimension 1 faces vanishes. We further have that the unbalanced faces of



$\mathcal{C}_1$  are precisely the ones in the polyhedral subspace  $D' := \mathcal{C}_1 \cap \Gamma_{P'}(W) = \mathcal{C}_2 \cap \Gamma_{P'}(W)$ . The facets of the polyhedral subspace  $D'$  are equipped with weight 1. Thus we obtain

$$\int_{\partial \mathcal{C}_1} \alpha \wedge l(\beta) = \int_{D'} \langle \alpha \wedge l(\beta); e_i \rangle_n.$$

Since  $l(\beta)|_{D'} = (\delta|_{D'})^*(\beta)$  and  $\delta(e_i) = 0$ , we have that  $\langle l(\beta); e_i \rangle_p|_{D'} = 0$  and therefore

$$\int_{D'} \langle \alpha \wedge l(\beta), e_i \rangle_n = (-1)^{n-p} \int_{D'} \langle \alpha \wedge l(\beta); e_i \rangle_p = (-1)^{n-p} \int_{D'} \langle \alpha; e_i \rangle_p \wedge l(\beta).$$

Altogether, we obtain

$$\int_V \alpha \wedge d''(l(\beta)) = (-1)^n \int_{D'} \langle \alpha; e_i \rangle_p \wedge l(\beta).$$

Denote by  $F : W \rightarrow \Gamma_{P'}(W)$  the map into the graph of the function  $P'$ . Then there exist polyhedral structures  $\mathcal{D}$  on  $D$  and  $\mathcal{D}'$  on  $D'$ , such that for each facet  $\sigma$  of  $\mathcal{D}$  the restriction  $F|_\sigma$  is linear, the image  $F(\sigma)$  is a facet of  $\mathcal{D}'$  and each facet of  $\mathcal{D}'$  is of this form. Then the inverse of  $F|_\sigma$  is given by  $\delta|_{\sigma'}$ . Thus  $\delta|_{\sigma'}$  is an isomorphism of rational polyhedra for each  $\sigma' \in \mathcal{D}'$ . Since we have that  $\delta^*$  preserves  $\langle \alpha; e_i \rangle_p$  and  $(\delta|_{D'})^*\beta = l(\beta)|_{D'}$  we obtain

$$\int_{D'} \langle \alpha; e_i \rangle_p \wedge l(\beta) = \int_D \langle \alpha; e_i \rangle_p \wedge \beta,$$

which concludes the proof.  $\square$

**Proposition 4.19.** *Let  $V \subset \mathbb{R}^r$  be a matroidal fan, then  $V$  has Poincaré duality.*

*Proof.* Let  $n$  be the dimension of  $V$ . We perform induction on  $r$ . The base case  $r = 0$  is obvious.

For the induction step, we have two cases, these being  $n = r$  and  $n < r$ . If  $n = r$ , then  $V = \mathbb{R}^n$  and we have

$$H^{p,q}(\mathbb{R}^n) = \Lambda^p \mathbb{R}^{n*} \otimes H^q(\mathbb{R}^n) \quad \text{and} \quad H_c^{p,q}(\mathbb{R}^n) = \Lambda^p \mathbb{R}^{n*} \otimes H_c^q(\mathbb{R}^n),$$

where  $H^q$ , respectively  $H_c^q$ , denote the usual de Rham cohomology. Thus we know  $H^{p,q}(\mathbb{R}^n) = 0$  and  $H_c^{n-p,n-q}(\mathbb{R}^n) = 0$  unless  $q = 0$ . Otherwise  $H^{p,0}(\mathbb{R}^n) = \Lambda^p \mathbb{R}^{n*}$  and  $H_c^{n-p,n}(\mathbb{R}^n) = \Lambda^{n-p} \mathbb{R}^{n*}$  and the PD map is just  $(-1)^p$  times the map induced by the canonical pairing  $\Lambda^p \mathbb{R}^{n*} \times \Lambda^{n-p} \mathbb{R}^{n*} \rightarrow \Lambda^n \mathbb{R}^{n*} \cong \mathbb{R}$ . Since this pairing is non-degenerate the PD map is an isomorphism.

If  $n < r$  then by Proposition 3.10 and Lemma 4.16 the only non-trivial case to check is when  $q = 0$ . In other words, that  $\text{PD} : H^{p,0}(V) \rightarrow H_c^{n-p,n}(V)^*$  is an isomorphism. Consider an open tropical modification  $\delta : V \rightarrow W$  along a divisor  $D$  where  $D, W \subset \mathbb{R}^{r-1}$  are matroidal fans. Now  $D$  and  $W$  have PD by the induction hypothesis, so that in the commutative diagram from Lemma 4.18 the vertical arrows on the left and right are isomorphisms. By the five lemma we obtain PD for  $V \subset \mathbb{R}^r$  and the proposition is proven.  $\square$

The next two lemmas help to prove Proposition 4.22, which is analogous to Proposition 4.19 but for spaces of the form  $V \times \mathbb{T}^r$  where  $V$  is a matroidal fan. We will relate the cohomologies of  $V \times \mathbb{T}^r$ ,  $V \times \mathbb{R}^r$  and  $V$  by way of an exact sequence.

**Lemma 4.20.** *Let  $Y = V \times \mathbb{T}^r$  for a polyhedral fan  $V$ . Then we have a short exact sequence*

$$0 \longrightarrow H^{p,0}(Y \times \mathbb{T}) \longrightarrow H^{p,0}(Y \times \mathbb{R}) \xrightarrow{\langle \cdot, e_i \rangle_p} H^{p-1,0}(Y) \longrightarrow 0$$

where  $e_i$  is the coordinate of  $\mathbb{R}$  in  $Y \times \mathbb{R}$ .

*Proof.* We use the explicit calculation in Proposition 3.16. First this shows that none of the cohomology groups in the statement change when we replace  $Y$  by  $V$ , thus we assume  $Y = V$ . After identifying  $\mathcal{L}^p$  with  $H^{p,0}$  and undualizing we have to show that

$$(2) \quad 0 \rightarrow \sum_{\sigma \in Y} \Lambda^{p-1} \mathbb{L}(\sigma) \xrightarrow{\wedge e_i} \sum_{\sigma \in Y} \Lambda^p \mathbb{L}(\sigma \times \mathbb{R}) \rightarrow \sum_{\sigma \in Y} \Lambda^p \mathbb{L}(\sigma) \rightarrow 0$$

is exact. The first map is clearly injective and the last map is clearly surjective. For exactness in the middle notice the composition of the maps is certainly zero and that any element  $v \in \sum_{\sigma \in Y} \Lambda^p \mathbb{L}(\sigma \times \mathbb{R})$  can be written  $v = \sum_{\sigma \in Y} v_\sigma + \left( \sum_{\sigma \in Y} v'_\sigma \right) \wedge e_i$  for  $v_\sigma \in \Lambda^p \mathbb{L}(\sigma)$  and  $v'_\sigma \in \Lambda^{p-1} \mathbb{L}(\sigma)$ . Now if  $v$  maps to zero then it is of the form  $\left( \sum_{\sigma \in Y} v'_\sigma \right) \wedge e_i$  and thus in the image of  $\wedge e_i$ . This proves exactness.  $\square$

**Lemma 4.21.** *Let  $Y = V \times \mathbb{T}^r$  of dimension  $n$ , where  $V \subset \mathbb{R}^s$  is a matroidal fan, then the following diagram*

$$\begin{array}{ccccc} H^{p,0}(Y \times \mathbb{T}) & \longrightarrow & H^{p,0}(Y \times \mathbb{R}) & \xrightarrow{\langle \cdot, e_i \rangle_p} & H^{p-1,0}(Y) \\ \downarrow \text{PD} & & \downarrow \text{PD} & & \downarrow (-1)^n \text{PD} \\ H_c^{n-p+1,n+1}(Y \times \mathbb{T})^* & \longrightarrow & H_c^{n-p+1,n+1}(Y \times \mathbb{R})^* & \xrightarrow{g^*} & H_c^{n-p+1,n}(Y)^*, \end{array}$$

which is obtained by the sequences in Proposition 4.15 and Lemma 4.20, commutes.

*Proof.* The proof follows exactly along the lines of the proof of the commutativity of the diagram in Lemma 4.18 for tropical modifications with  $Y$  replacing  $D$ ,  $Y \times \mathbb{R}$  replacing  $V$  and  $Y \times \mathbb{T}$  replacing  $\bar{V}$  and  $P'$  being any constant function.  $\square$

**Proposition 4.22.** *Let  $Y = V \times \mathbb{T}^r$  for a matroidal fan  $V \subset \mathbb{R}^s$ . Then  $Y$  has Poincaré duality.*

*Proof.* We do induction on  $r$  with  $r = 0$  being Proposition 4.19. For the induction step we have to show that if  $Y$  has PD then  $Y \times \mathbb{T}$  also has PD. Since  $Y$  is a basic open subset,  $H^{p,q}(Y) = 0$  unless  $q = 0$  by Proposition 3.10. Since  $Y$  has PD, we have  $H_c^{p,q}(Y) = 0$  unless  $q = n = \dim(Y)$ . Note also that  $Y \times \mathbb{R} = V \times \mathbb{R} \times \mathbb{T}^r$  and so this space has PD. We therefore also have  $H_c^{p,q}(Y \times \mathbb{R}) = 0$  unless  $q = n + 1 = \dim(Y \times \mathbb{R})$ . Now the sequence from Proposition 4.15 yields that  $H_c^{p,q}(Y \times \mathbb{T}) = 0$  if  $q \neq n, n + 1$  and that

$$(3) \quad 0 \rightarrow H_c^{p,n}(Y \times \mathbb{T}) \rightarrow H_c^{p,n}(Y) \xrightarrow{f} H_c^{p,n+1}(Y \times \mathbb{R}) \rightarrow H_c^{p,n+1}(Y \times \mathbb{T}) \rightarrow 0$$

is exact. By the commutativity of the second square of the diagram in Lemma 4.21, the map  $f$  is up to sign the dual map to  $\langle \cdot, e_i \rangle_p$ , once we use PD for  $Y$  and  $Y \times \mathbb{R}$  to identify  $H^{p,0}(Y \times \mathbb{R}) \cong H_c^{n-p+1,n+1}(Y \times \mathbb{R})^*$  and  $H^{p-1,0}(Y) \cong H_c^{n-p+1,n}(Y)^*$ . Now  $\langle \cdot, e_i \rangle_p$  is known to be surjective by Lemma 4.20, thus  $f$  is injective and we have  $H_c^{p,q}(Y \times \mathbb{T}) = 0$  unless  $q = n + 1$ . Since  $Y \times \mathbb{T}$  is a basic open subset, we also know  $H^{p,q}(Y \times \mathbb{T}) = 0$  unless  $q = 0$  by Proposition 3.10, thus we only have to consider  $\text{PD} : H^{p,0}(Y \times \mathbb{T}) \rightarrow H_c^{n+1-p,n+1}(Y \times \mathbb{T})$ . Note that this is precisely the first vertical map in the diagram in Lemma 4.21 and that the respective first horizontal maps are injective by Lemma 4.20 and the sequence (3). Since the other vertical maps are isomorphisms, this shows that  $Y \times \mathbb{T}$  has PD.  $\square$

Before we complete the proof of Theorem 2, we make the following observation.

**Remark 4.23.** By Proposition 3.16 and Proposition 3.10 we have  $H^{p,q}(V \times \mathbb{T}^r) = H^{p,q}(V)$ . Since Poincaré duality holds for these spaces, we further have  $H_c^{p,q}(V) = H_c^{p+r,q+r}(V \times \mathbb{T}^r)$ . This is the same behaviour that taking the product with  $\mathbb{C}$  exhibits for classical Dolbeault cohomology.

The following technical lemma allows us to deduce Poincaré duality for basic open subsets of matroidal fans.

**Lemma 4.24.** *Let  $Y = V \times \mathbb{T}^r \subset \mathbb{T}^{s+r}$  for a matroidal fan  $V \subset \mathbb{R}^s$  and  $\Omega$  a basic open neighborhood of  $(0, \dots, 0, -\infty, \dots, -\infty)$ . Then there are canonical isomorphisms*

$$H^{p,q}(Y) \rightarrow H^{p,q}(\Omega) \text{ and } H_c^{p,q}(\Omega) \rightarrow H_c^{p,q}(Y)$$

*which are induced by restriction and inclusion of superforms. In particular  $\Omega$  has PD.*

*Proof.* For  $H^{p,q}$  this follows immediately from the explicit calculation we did in Proposition 3.16. For cohomology with compact support, we first see that there is a homeomorphism between  $U$  and  $Y$  which respects strata and polyhedra. This induces an isomorphism of compactly supported tropical cohomology and thus we already know  $H_c^{p,q}(\Omega) \cong H_c^{p,q}(Y)$  by Theorem 3.18. By PD for  $Y$  we furthermore know that these cohomology groups are finite dimensional, thus it is sufficient to show that inclusion of superforms induces a surjective map on cohomology. Again by PD for  $Y$  this is trivial if  $q \neq n := \dim Y$ . We choose a basis  $\alpha_1, \dots, \alpha_k$  of  $H^{p,0}(Y)$ . By PD for  $Y$  there exist  $\omega_1, \dots, \omega_k \in H_c^{n-p,n}(Y)$  such that  $\int_Y \alpha_i \wedge \omega_j = \delta_{ij}$  and for surjectivity of  $H_c^{n-p,n}(\Omega) \rightarrow H_c^{n-p,n}(Y)$  it is sufficient to show that there exist  $\beta_1, \dots, \beta_k \in H_c^{n-p,n}(\Omega)$  such that  $\int_Y \alpha_i \wedge \beta_j \neq 0$  if and only if  $i = j$ . Let  $B$  be the union of the supports of all  $\omega_i$  and  $C \in \mathbb{R}_{>0}$  and  $v \in \mathbb{R}^r$  such that  $B \subset C \cdot \Omega + v$ . Define  $F$  to be the extended affine map given by  $w \mapsto C \cdot w + v$  and set  $\beta_i := F^*(\omega_i) \in \mathcal{A}_c^{p,q}(\Omega)$  for all  $i$ . Since  $\alpha_i \in \mathcal{L}^p(Y)$  we have  $F^*(\alpha_i) = C^p \alpha_i$  and thus we have

$$\int_Y \alpha_i \wedge \beta_j = \int_Y \alpha_i \wedge F^*(\omega_j) = C^{-p} \int_Y F^*(\alpha_i \wedge \omega_j) = C^{n-p} \delta_{ij},$$

where the last equality is given by the transformation formula [Gub13, 2.4]. This proves surjectivity and thus the lemma.  $\square$

**Lemma 4.25.** *Let  $V$  be a matroidal fan in  $\mathbb{R}^s$ ,  $Y = V \times \mathbb{T}^r$  and  $\Omega$  a basic open subset of  $Y$ . Then  $\Omega$  has PD.*

*Proof.* If  $I \subset [r]$  is the maximal sedentarity among points of  $\Omega$ , then  $\Omega$  is a basic open subset of  $V \times \mathbb{T}^{|I|} \times \mathbb{R}^{r-|I|}$  of maximal sedentarity. Let  $x$  be a point in the relative interior of the minimal face of the basic open set  $\Omega$ . The star of any point in a matroidal fan is again a matroidal fan, see [AK06, Proposition 2]. Applying this fact to the fan  $V \times \mathbb{R}^{r-|I|}$  we obtain that, after translation of  $x$  to the origin,  $\Omega$  is a basic open neighborhood of  $(0, \dots, 0, -\infty, \dots, -\infty)$  in the matroidal fan  $\text{Star}_x(V) \times \mathbb{T}^{|I|}$ .

Thus we are now in the situation of Lemma 4.24, which shows that  $\Omega$  has PD.  $\square$

Altogether, we now obtain Poincaré duality for tropical manifolds:

**Theorem 4.26.** *Let  $X$  be an  $n$ -dimensional tropical manifold. Then the Poincaré duality map is an isomorphism for all  $p, q$ .*

*Proof.* We write  $\mathcal{A}_c^{p,q*}$  for the sheaf  $U \mapsto \text{Hom}_{\mathbb{R}}(\mathcal{A}_c^{p,q}(U), \mathbb{R})$ . Then  $\mathcal{A}_c^{p,q*}$  is a sheaf, since  $\mathcal{A}^{p,q}$  is fine. Furthermore  $\mathcal{A}_c^{p,q*}$  is a flasque sheaf, since for  $U' \subset U$  the inclusion  $\mathcal{A}_c^{p,q}(U') \rightarrow \mathcal{A}_c^{p,q}(U)$  is injective. We then obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{L}^p & \longrightarrow & \mathcal{A}^{p,0} & \xrightarrow{d''} & \mathcal{A}^{p,1} \longrightarrow \dots \\ & & \downarrow \text{id} & & \downarrow \text{PD} & & \downarrow \text{PD} \\ 0 & \longrightarrow & \mathcal{L}^p & \longrightarrow & \mathcal{A}_c^{n-p,n*} & \xrightarrow{d''^*} & \mathcal{A}_c^{n-p,n-1*} \longrightarrow \dots \end{array}$$

and we have

$$H^q(\mathcal{A}_c^{n-p,n-\bullet*}(U), d''^*) = (H_q(\mathcal{A}_c^{n-p,n-\bullet}(U), d''))^* = H_c^{p,q}(U)^*.$$

If we consider the sections of this diagram over a basic open subset, then the first row is exact by Proposition 3.10. By Lemma 4.25 the second row is then also exact. This shows that both rows are exact sequences of sheaves on  $X$ . Thus we have a commutative diagram of

acyclic resolutions of  $\mathcal{L}^p$ , thus PD induces isomorphisms on the cohomology of the complexes of global sections, which precisely means that  $X$  has PD.  $\square$

When  $X$  is a compact tropical manifold, the above theorem immediately implies the following.

**Corollary 4.27.** *Let  $X$  be a compact tropical manifold of dimension  $n$ . Then*

$$\text{PD} : H^{p,q}(X) \rightarrow H^{n-p,n-q}(X)^*$$

*is an isomorphism for all  $p, q$ .*

## REFERENCES

- [AB] Karim Alexander Adiprasito and Anders Björner. Filtered geometric lattices and lefschetz section theorems over the tropical semiring. <http://arxiv.org/abs/1401.7301>.
- [AHK] Karim Alexander Adiprasito, June Huh, and Eric Katz. Hodge theory for combinatorial geometries. <http://arxiv.org/abs/1511.02888>.
- [AK06] Federico Ardila and Caroline Klivans. The Bergman complex of a matroid and phylogenetic trees. *J. Comb. Theory Ser. B*, 96(1):38–49, 2006.
- [BIMS15] Erwan Brugallé, Ilia Itenberg, Grigory Mikhalkin, and Kristin Shaw. Brief introduction to tropical geometry. In *Gökova Geometry-Topology conference*, 2015. To appear in Proceedings of 21st Gökova Geometry-Topology Conference.
- [Bre97] Glen E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.
- [CLD12] Antoine Chambert-Loir and Antoine Ducros. Formes différentielles réelles et courants sur les espaces de Berkovich. 2012. <http://arxiv.org/abs/1204.6277>.
- [EH00] David Eisenbud and Joe Harris. *The geometry of schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.
- [FS05] Eva Maria Feichtner and Bernd Sturmfels. Matroid polytopes, nested sets and Bergman fans. *Port. Math. (N.S.)*, 62(4):437–468, 2005.
- [GK14] Walter Gubler and Klaus Künnemann. A tropical approach to non-archimedean Arakelov theory. 2014. <http://arxiv.org/abs/1406.7637>.
- [Gub12] Walter Gubler. A guide to tropicalization. 2012. <http://arxiv.org/abs/1108.6126>.
- [Gub13] Walter Gubler. Forms and currents on the analytification of an algebraic variety (after Chambert-Loir and Ducros). 2013. <http://arxiv.org/abs/1303.7364>.
- [IKMZ] Ilia Itenberg, Ludmil Khazarkov, Grigory Mikhalkin, and Ilia Zharkov. Tropical homology. In preparation.
- [Ive86] Birger Iversen. *Cohomology of sheaves*. Universitext. Springer-Verlag, Berlin, 1986.
- [Jel15] Philipp Jell. A Poincaré lemma for real-valued differential forms on Berkovich spaces. *Mathematische Zeitschrift*, 2015. available at: <http://dx.doi.org/10.1007/s00209-015-1583-8>.
- [Lag12] Aron Lagerberg. Super currents and tropical geometry. *Math. Z.*, 270(3-4):1011–1050, 2012.
- [Mik06] Grigory Mikhalkin. Tropical geometry and its applications. In *International Congress of Mathematicians. Vol. II*, pages 827–852. Eur. Math. Soc., Zürich, 2006.
- [MR] Grigory Mikhalkin and Johannes Rau. Tropical geometry. Draft of a book available at: <https://www.dropbox.com/s/9lpv86oz5f4za75/main.pdf>.
- [MZ13] Grigory Mikhalkin and Ilia Zharkov. Tropical eigenwave and intermediate jacobians. 2013. arXiv:1303.3255v2.
- [OT92] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*. Springer Verlag, 1992.
- [Oxl11] James Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.
- [Ram05] Sundararaman Ramanan. *Global calculus*, volume 65 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.
- [Sha11] Kristin Shaw. Tropical intersection theory and surfaces, 2011. PhD Thesis, available at <http://www.math.toronto.edu/shawkm/>.
- [Sha13a] Kristin Shaw. Tropical  $(1, 1)$ -homology for floor decomposed surfaces. In E. Brugallé, M. A. Cueto, A. Dickenstein, E.M. Feichtner, and I. Itenberg, editors, *Algebraic and Combinatorial Aspects of Tropical Geometry*, volume 589, pages 529–550, Providence, RI, 2013. American Mathematical Society.
- [Sha13b] Kristin Shaw. A tropical intersection product in matroidal fans. *SIAM J. Discrete Math.*, 27(1):459–491, 2013.
- [Sha15] Kristin Shaw. Tropical surfaces. 2015. <http://arxiv.org/abs/1506.07407>.
- [Stu02] Bernd Sturmfels. *Solving systems of polynomial equations*, volume 97 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, RI., 2002.

- [War83] Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition.
- [Wel80] Raymond O. Wells, Jr. *Differential analysis on complex manifolds*, volume 65 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, second edition, 1980.
- [Zha13] Ilia Zharkov. The Orlik-Solomon algebra and the Bergman fan of a matroid. *J. Gökova Geom. Topol. GGT*, 7:25–31, 2013.

PHILIPP JELL, UNIVERSITÄT REGENSBURG, UNIVERSITÄTSSTRASSE 31, 93053 REGENSBURG, GERMANY

*E-mail address:* philipp.jell@ur.de

KRISTIN SHAW, TECHNISCHE UNIVERSITÄT BERLIN, MA 6-2, 10623 BERLIN, GERMANY.

*E-mail address:* shaw@math.tu-berlin.de

JASCHA SMACKA, UNIVERSITÄT REGENSBURG, UNIVERSITÄTSSTRASSE 31, 93053 REGENSBURG, GERMANY

*E-mail address:* jascha.smacka@ur.de